Alg Number Theory (621) / Problem Set 3

1) Let $K = \mathbb{Q}_p$ and $L = K(\zeta_n)$, $n = p^e$, be the $n^{th}$ cyclotomic field of over $\mathbb{Q}_p$.
   a) Compute the Herbrand mappings $\varphi_{L/K}$ and $\psi_{L/K}$.
   b) Deduce from this the higher ramification groups in the upper numbering $G^t$ of $G = \text{Gal}(L/K)$.

2) Prove the local Kronecker–Weber Theorem in the case $p = 2$:
   Each finite Abelian extension of $\mathbb{Q}_2$ is contained in some cyclotomic extension $\mathbb{Q}_2(\zeta_n)$.
   What is the maximal Abelian tamely ramified extension of $\mathbb{Q}_2$?

Adeles and Ideles

3) Let $K|\mathbb{Q}$ be some number field, $\mathbb{A}_K$ its ring of adeles. Using the fact that $(\mathbb{A}_K, +)/(K, +)$ is a compact group, prove the following:

   Generalized Approximation Lemma. Let the following be given: $\epsilon > 0$, a place $p_0$, a finite set of places $S$ with $p_0 \notin S$ but containing all the archimedean places, and $x_p \in K_p$ for each $p \in S$. Then there exists $x \in K$ such that $|x - x_p|_p < \epsilon \forall p \in S$, and $|x|_p \leq 1 \forall p \in S, p \neq p_0$.

4) Show the following facts about the adeles of $\mathbb{Q}$:
   a) $\mathbb{A}_\mathbb{Q} \cong \mathbb{R} \times (\hat{\mathbb{Z}}, \mathbb{Q})$ canonically.
   b) $(\mathbb{R} \times \hat{\mathbb{Z}}, +)/(\mathbb{Z}, +)$ is a compact connected group, and at the same time a $\mathbb{Q}$ vector space.
   c) Describe the idele class group $\mathbb{C}_\mathbb{Q}$.

5) Let $K$ be a number field, $\mathcal{O}$ its ring of integers, and $\mathfrak{m}$ some module of $K$.
   a) Let $\mathcal{O}_+^\times$ and $\mathcal{O}_+^\mathfrak{m}$ denote the positive units, resp. the positive $\mathfrak{m}$ units of $K$. Show that the following canonical sequence is exact: $1 \rightarrow \mathcal{O}_+^\times/\mathcal{O}_+^\mathfrak{m} \rightarrow (\mathcal{O}/\mathfrak{m})^\times \rightarrow \mathcal{C}_K^\mathfrak{m} \rightarrow \mathcal{C}_K^1 \rightarrow 1$
   b) Describe the kernels of the canonical maps $\mathcal{C}_K^\mathfrak{m} \rightarrow \mathcal{C}_K$, respectively $\mathcal{C}_K^\mathfrak{m} \rightarrow \mathcal{C}_K^n$ for $n|\mathfrak{m}$.

6) Let $L|K$ be a finite Galois extension of number fields, and $\tilde{\mathcal{I}}_{L/K} : \mathcal{C}_L \rightarrow \mathcal{C}_L$ the canonical mapping induced by the inclusion $\mathcal{I}_{L/K}$. Prove or disprove:
   a) $\tilde{\mathcal{I}}_{L/K} : \mathcal{C}_L \rightarrow \mathcal{C}_L$ is injective.
   b) $\tilde{\mathcal{I}}_{L/K}(\mathcal{C}_L) = (\mathcal{C}_L)^G$

7) Let $\mathfrak{D}$ be a finite dimension division algebra over a complete valued field field $(K, | |)$. Show the following:
   a) $| |$ has a unique prolongation $| |_{\mathfrak{D}}$ to $\mathfrak{D}$; and $| |_{\mathfrak{D}}$ coincides on every subfield $L \subset \mathfrak{D}$ with the unique prolongation $| |_{L}$ of $| |$ to $L$.

Next suppose that $| |$ is n.a. As in the usual commutative case define the “valuation ring” $\mathfrak{R}$ of $| |_{\mathfrak{D}}$ and is “valuation ideal” $\mathfrak{M} \subset \mathfrak{R}$. Show the following:
   b) $\mathfrak{R}$ is an $R_1$ $|$-algebra, $\mathfrak{M} \subset \mathfrak{R}$ is a left and right ideal of $\mathfrak{R}$, and the factor ring $\mathfrak{R}/\mathfrak{M}$ is a skew field.
   (Thus $\mathfrak{M}$ is a maximal ideal.)
   c) Moreover, if $| |$ is discrete, then $| |_{\mathfrak{D}}$ is discrete too, and defining $e := e(\mathfrak{D}|K)$ and $f := f(\mathfrak{D}|K)$ as usual (HOW?), one has $\dim_K \mathfrak{D} = ef$.
   d) Finally, if $K$ is a locally compact n.a. field, and $K$ is the center of $\mathfrak{D}$, then $e = f$, and $\mathfrak{t}$ is a finite field.