1) Prove the following assertions from the Lecture:
   a) Let $R$ be a reduced commutative ring having finitely many minimal prime ideals denoted $\{p_\alpha\}_\alpha$, and set $R_\alpha = R/p_\alpha$. Then $\text{Quot}(R) \cong \oplus_\alpha \text{Quot}(R_\alpha)$, and $\tilde{R} \cong \oplus_\alpha \tilde{R}_\alpha$ canonically.
   b) If $X$ is a $k$-prevariety with irreducible components $X_\alpha$, and $\tilde{\phi}_\alpha : \tilde{X}_\alpha \to X_\alpha$ is the normalization of $X_\alpha$, then the canonical morphism $\coprod_\alpha \tilde{\phi}_\alpha : \coprod_\alpha \tilde{X}_\alpha \to X$ is the normalization of $X$.
   c) The normalization morphism $\tilde{\phi} : \tilde{X} \to X$ is an affine morphism, i.e., if $U \subset X$ is an open affine subset, then its preimage $\tilde{\phi}^{-1}(U)$ is an affine subset of $\tilde{X}$.

2) Prove the following:

**Theorem.** Let $X$ be a $k$-prevariety with ring of rational functions $k(X)$. Let $\iota : k(X) \hookrightarrow K$ be a reduced finite $k(X)$-algebra. Then there exists a normal $k$-prevariety $X'$ with $k(X') = K$ together with a dominant $k$-morphism $\phi' : X' \to X$ with the following properties:
   (i) $\phi'$ is finite, surjective, and $\iota = \phi'^\#$.
   (ii) Universal property: Given a dominant $k$-morphism $\psi : Y \to X$ and a factorization of the resulting $k$-embedding $\psi^\# : k(X) \to k(Y)$ through $\iota$, say of the form $\psi^\# : k(X) \hookrightarrow K \twoheadrightarrow k(Y)$, then there exists a unique $\psi' : Y \to X'$ s.t. $\psi := \phi' \circ \psi'$, and $\iota = \psi'^\#$.

In the above context, $\phi' : X' \to X$ is called the **normalization of $X$ in the extension** $\iota : k(X) \hookrightarrow K$.

3) Let $k = \mathbb{F}_{p}(t)$, $p = 2, 3$, be the rational function field in the variable $t$ over the prime field $\mathbb{F}_p$. Consider the closed $k$-subvariety $Y = V(Y_1^2 - Y_2 - tY_1^9) \subset \mathbb{A}_k^2$, and let $X$ be its projective closure in $\mathbb{P}_k^2$.
   a) Prove or disprove: $Y$ is (absolutely) irreducible. The same question for $X$.
   b) Describe the normal, resp. regular, resp. smooth points of $Y$. The same question for $X$.
   c) Describe the normalizations $\tilde{Y} \to Y$ and $\tilde{X} \to X$.
   d) Is $\tilde{Y}$, respectively $\tilde{X}$, geometrically normal and/or geometrically regular?

4) The same problem for $Y = V(X_1^2X_2^3X_3^3 - X_2^2X_3^3 - 1)$, but over an arbitrary base field $k$, $\text{char}(k) \neq 2$.

**Hint:** Set $k[Y] = k[x_1, x_2, x_3]$. Then $\Omega_{Y/k}$ is freely generated by $dx_1, dx_3$ (WHY?), etc.

5) Let $X$ and $Y$ be $k$-prevarieties, $X \times_k Y$ their product. For $x \in X$, $y \in Y$, and $(x, y) \in X \times_k Y$, prove or disprove:
   a) $x$ and $y$ are normal (resp. regular) points if $(x, y)$ is a normal (resp. regular) point.
   b) The same question with “smooth” in stead of “normal”.

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Math 624 (Algebraic Geometry) / Problem Set 7

Due: Dec 17, 2004