1) Answer the following:
   a) Find a g.c.d. of the polynomials \( f(X) = X^8 - X^7 + 2X - 6 \) and \( g(X) = x^4 + 3X \) in \( \mathbb{F}_p[X] \), where \( p \) is any prime number.
   b) Show that the ring \( R = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Z} \} \) is Euclidean w.r.t. \( \varphi : R \to \mathbb{N} \), \( \varphi(a + b\sqrt{2}) = |a^2 - 2b^2| \).
      Find a g.c.d. of \( x = 7 + 3\sqrt{2} \) and \( y = 5 + 6\sqrt{2} \).

2) Let \( R = \mathbb{F}_2[X, Y] \) be the polynomial ring over \( \mathbb{F}_2 \).
   a) Show that the ideal \( \mathfrak{a} = (X - Y) R + (X - 1) R \) in \( R \) is maximal and not principal.
   b) Prove or disprove: All the ideals \( \mathfrak{a} \) of \( R \) which contain \( X + 1 \) are of the form \( f(Y) R + (X + 1) R \), where \( f(Y) \) is an irreducible polynomial; and \( \mathfrak{a} \) is a maximal ideal of \( R \) iff \( f(Y) \) is an irreducible polynomial.

3) Let \( f : N \to M \) be an \( R \)-homomorphism of \( R \)-modules.
   a) Prove that \( \ker(f) \) and \( \text{im}(f) \) are \( R \)-submodules of \( N \).
   b) Suppose that \( f \) is surjective. Prove that if \( (x_i)_{i \in I} \) is a system of generators of \( N \) as an \( R \)-module, then \( (f(x_i))_{i \in I} \) is a system of generators of \( M \).

4) For an \( R \)-module \( M \) we denote \( \text{Ann}_R(M) = \{ a \in R \mid a x = 0 \forall x \in M \} \).
   a) Find \( \text{Ann}_R(M) \) in the case \( R = \mathbb{Z} \) and \( M = \mathbb{Z} \times \mathbb{Z}/15 \times \mathbb{Z}/25 \).
   b) Same question in case \( M = R/a \times R/b \), where \( a \) and \( b \) are ideals of \( R \).

5) All the matrices below are from \( M_2(R) \), where \( R \) is a commutative ring with 1. Prove or disprove:
   a) Let \( A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \), with \( a \in R^\times \) a unit of \( R \). Then for \( B = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \) there exist matrices \( X, Y \in M_2(R) \)
      s.t. \( B = XAY \).
   b) For every non-zero matrix \( A \in M_2(R) \) one has: \( M_2(R) = M_2(R) \cdot A \cdot M_2(R) \), provided at least one of
      the coefficients of \( A \) is a unit in \( R \).

6) Let us denote \( V = \mathbb{R}^3 \) viewed as an \( \mathbb{R} \)-vector space endowed with the “canonical basis” \( \mathcal{E} = (e_1, e_2, e_3) \).
   a) Find an \( \mathbb{R} \)-basis for the \( \mathbb{R} \)-subspace of \( V \) generated by the vectors \( v_1 = (3, 1, 1), v_2 = (1, -2, 0), v_3 = (7, 5, 3) \).
   b) Define \( \varphi : V \to V \) by \( \varphi(e_i) = v_i \ (i = 1, 2, 3) \). Describe \( \ker(\varphi) \), and prove or disprove: \( \varphi \) is surjective.

7) Let us denote \( V = \mathbb{R}^3 = \{ (x_1, x_2, x_3) \mid x_i \in \mathbb{R} \} \) viewed as an \( \mathbb{R} \)-vector space.
   a) Prove that \( \mathcal{A} = (u_1, u_2, u_3) \) with \( u_1 = (1, 2, 2), u_2 = (2, 1, 2), u_3 = (3, 1, 1), \) is an \( \mathbb{R} \)-basis of \( V \).
   b) Let \( \varphi : V \to V \) be the reflexion w.r.t. the plane \( x_1 = 0 \) in \( V \), i.e.: \( \varphi(x_1, x_2, x_3) = (-x_1, x_2, x_3) \). Find
      the matrix of \( \varphi \) in the basis \( \mathcal{A} \).

8) Consider \( A = \begin{pmatrix} 0 & 1 & 1 & 3 \\ 3 & 0 & 1 & 2 \end{pmatrix} \in M_{2 \times 4}(\mathbb{R}) \). Let \( V \subset M_{4 \times 1}(\mathbb{R}) \) be the set of all column vectors \( X \) such \( AX = 0 \).
   a) Show that \( V \) is an \( \mathbb{R} \)-subspace of \( M_{4 \times 1}(\mathbb{R}) \).
   b) Find an \( \mathbb{R} \)-basis of \( V \).