Lecture 3 at HUJI 2009-04-01

Recall def'ns of $\mathcal{T}(M)$, $\mathcal{S}$, l_*, I_*, $\pi : [0,\infty)^{\mathcal{S}} \to \mathbb{P}([0,\infty)^{\mathcal{S}})$.

$\mathrm{Mod}(M)$ acts on $\mathcal{S}$ in the obvious way: $\varphi \cdot c := \varphi(c)$.

Then $\mathrm{Mod}(M) \curvearrowright [0,\infty)^{\mathcal{S}}$ as: $(\varphi \cdot f)(c) = f(\varphi^{-1} \cdot c)$.

With this, l_* and I_* are equivariant. Let's check l_*.

$$l_*(\varphi \cdot (X,m)) = l_*((X, m \circ \varphi^{-1})) = \{ c \longmapsto l_c(X, m \circ \varphi^{-1}) \}$$

$$= \{ c \longmapsto l_{\varphi^{-1}(c)}(X,m) \} = \varphi \cdot l_*((X,m)).$$

Recall l_* is an embedding. This means a surface is determined by the lengths of its scc's. $\pi \circ l_*$ is also an embedding, meaning it's impossible in hyperbolic geometry to simultaneously expand all the scc's by the same factor.

To better understand I_*, let's begin with intersections of scc's.

$$i : \mathcal{S} \times \mathcal{S} \longrightarrow [0,\infty)$$
$$(b, c) \longmapsto i(b,c)$$

is the $\min\limits_{\substack{b' \sim b \\ c' \sim c}} \# |b' \cap c'|$.

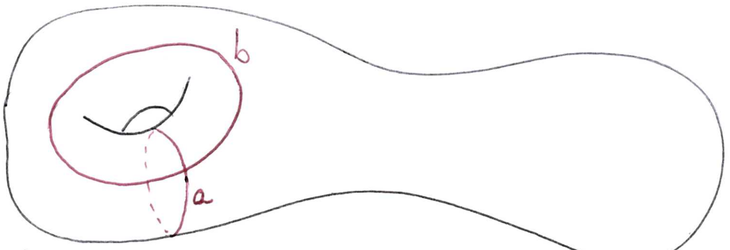
$\sim$ means isotopic $\longrightarrow$

FACT: If b' & c' are closed geods in some hyper. metric on M then
$i(b,c) = \# |b' \cap c'|$.

With this we can define $i_* : \mathcal{S} \longrightarrow [0,\infty)^{\mathcal{S}}$ & $I_* := \pi \circ i_*$.

$i_*(\mathcal{S}) \subset \mathbb{P}([0,\infty)^{\mathcal{S}})$ has a slightly surprising topology.

Claim: $\bar{i}_*(D_a^n(b)) \longrightarrow \bar{i}_*(a)$.

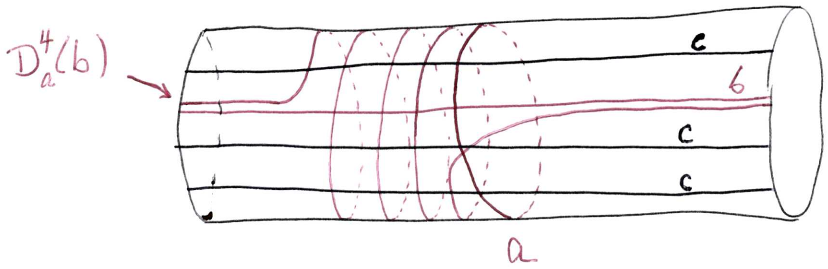


Pf: Consider $\frac{1}{n} i_*(D_a^n(b)) \subseteq [0,\infty)^{\mathcal{S}}$.

Pick $c \in \mathcal{S}$. Suppose $i(a,c) = 0$.

Then $i(\frac{1}{n} D_a^n(b), c) = \frac{1}{n} i(b, D_a^{-n}(c)) = \frac{1}{n} i(b,c) \to 0$.

Next assume $i(a,c) = k > 0$. Examine a small annular nbd. of a.

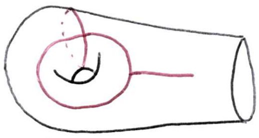


in the picture $k=3$

$$\frac{1}{n} i(D_a^n(b), c) \leq \underbrace{\frac{1}{n} i(b,c)}_{\text{intersections outside the annulus about } a} + \underbrace{\frac{1}{n} \cdot i(na, c)}_{\text{intersections inside the annulus}} \longrightarrow k.$$

$$\therefore \quad \frac{1}{n} i_*(D_a^n(b)) \longrightarrow i_*(a) \quad \Rightarrow \quad \bar{i}_*(D_a^n(b)) \longrightarrow \bar{i}(a). \qquad \boxtimes$$

We'll next describe an embedding $\mathcal{S} \to \mathcal{MF}$. Pick $c \in \mathcal{S}$. Choose a min'l graph $G \subset (M-c)$ such that $(M-c)$ is homeom. to a small neigh. of G in $M-c$. (Minimality ensures there are no spurious edges, e.g.



is not allowed.)

Then $M-G$ is an annulus. Choose a homeom. $(M-G) \longrightarrow [0,1] \times S^1$ taking c to $\{\frac{1}{2}\} \times S^1$. Use the homeo to pull back the "vertical" foliation of the annulus to $M-G$, so c is a closed leaf. Pull back the transverse measure of $[0,1]$. Add the singular leaves G to obtain a measured singular foliation $\mathcal{F}_c$. The choice of G was not canonical.

Prop: All choices of G result in Whitehead equivalent measured singular foliations.

(Pf in F-L-P 5.III.)

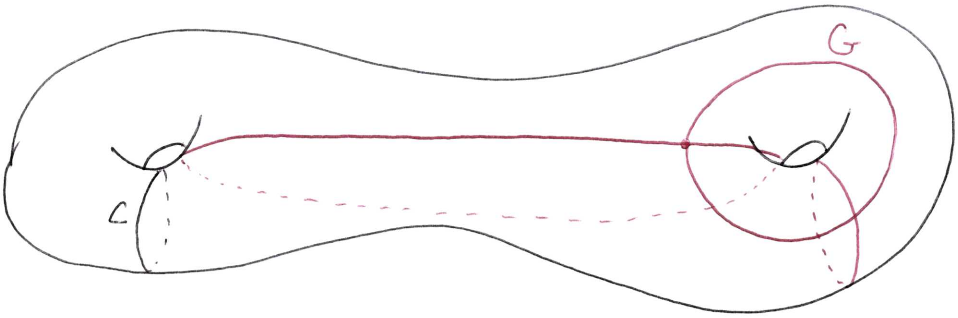
We therefore have a set map $\mathcal{S} \longrightarrow \mathcal{MF}$.

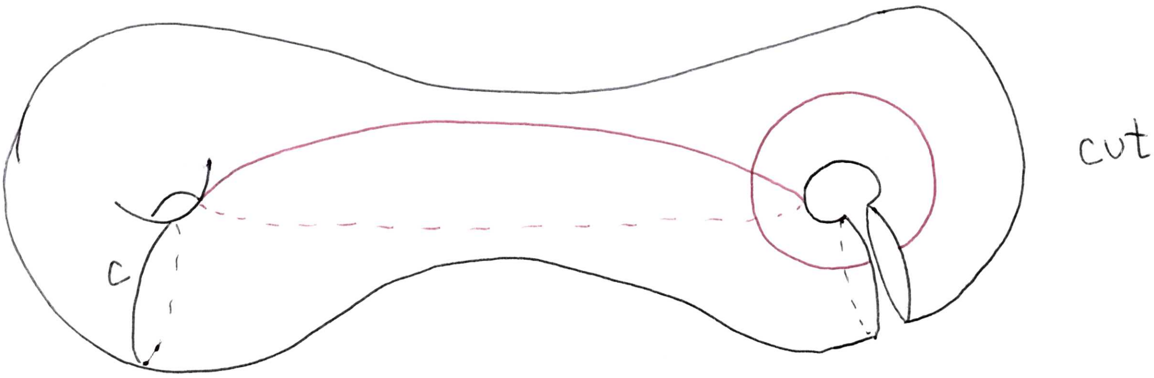
It's not too hard to believe this map is injective. In fact, we have the following commutative diagram of injections.

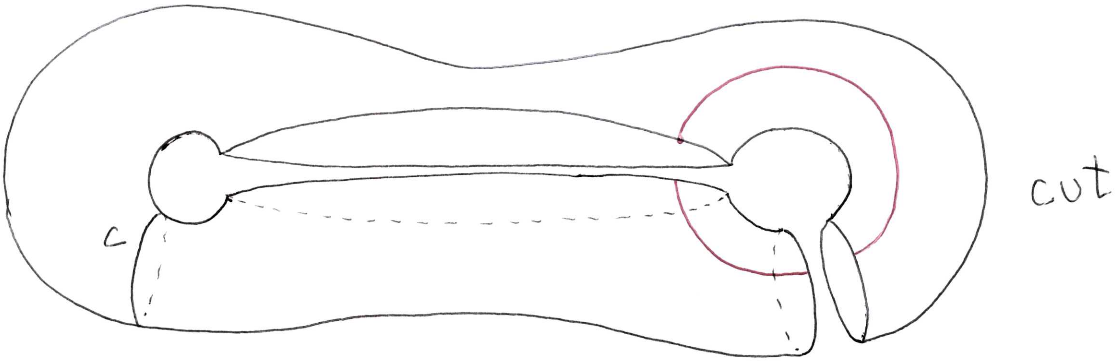
$$\begin{tikzcd}
\mathcal{S} \arrow[r, "i_*"] \arrow[d] \arrow[dr, phantom, "\circ"] & {[0,\infty)^{\mathcal{S}}} \\
\mathcal{MF} \arrow[ur, "I_*"'] &
\end{tikzcd}$$

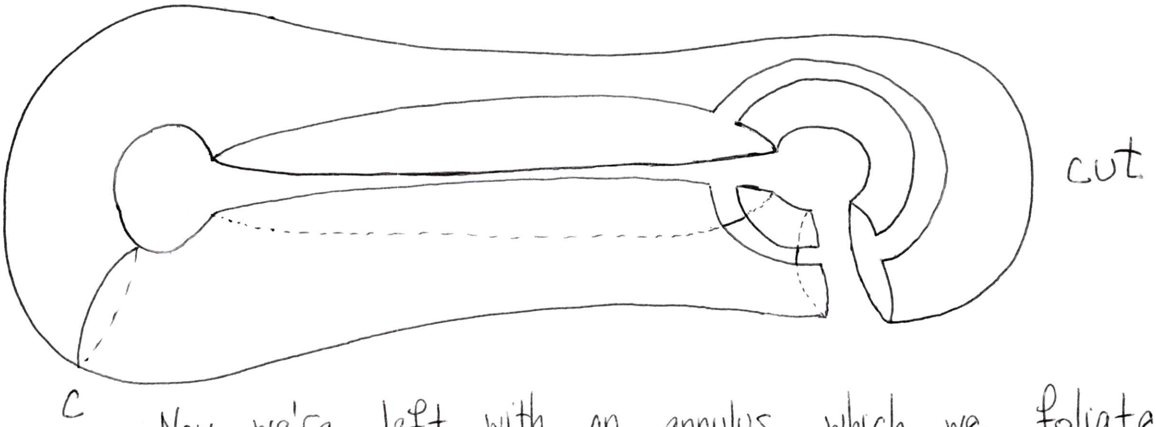
[FACT: $i_*(\mathcal{S}) \subset (\pi \circ I_*)(\mathcal{MF})$ is dense. Up to scaling, any meas. singular foliation is near a simple closed curve.]

Here are some pictures explaining how to get $\mathcal{F}_c$ from c.

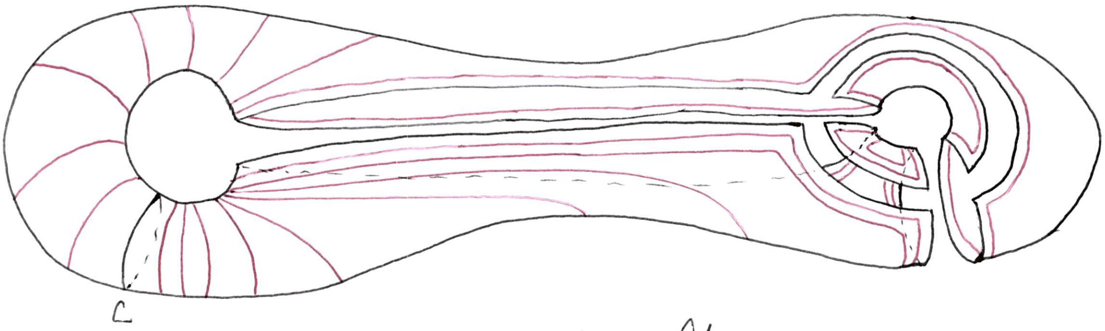




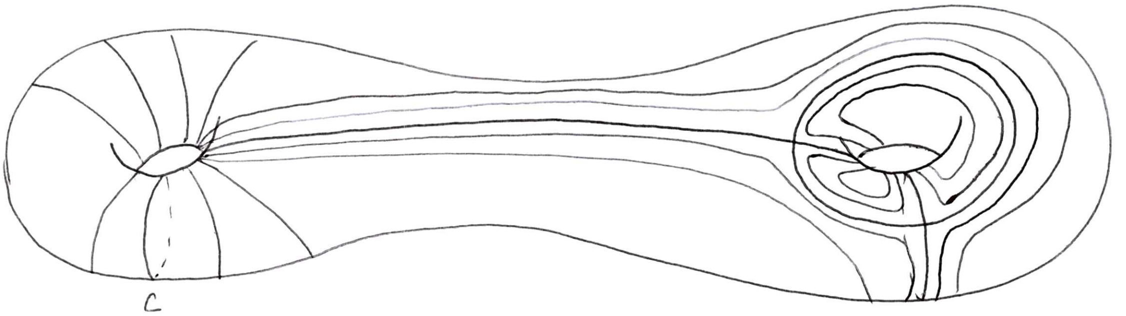




Now we're left with an annulus, which we foliate.



Glue up to obtain $\mathcal{F}_c$



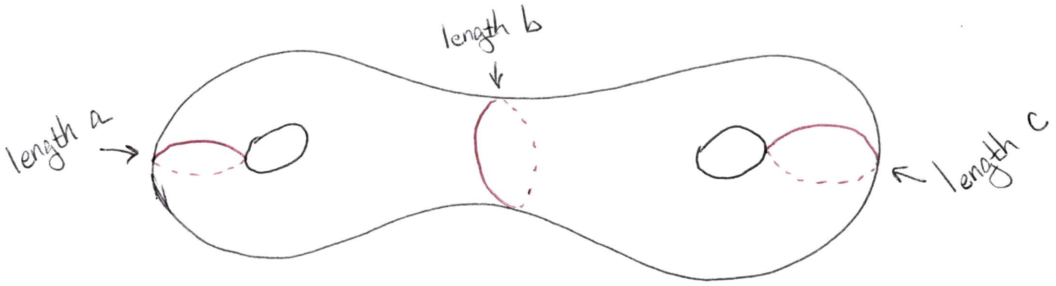
Claim: $\mathcal{S} \hookrightarrow \mathbb{P}\mathcal{M}\mathcal{F} \subset \mathbb{P}([0,\infty)^{\mathcal{S}})$ has dense image, i.e. for measured singular foliation $\mathcal{F}$ $\exists$ sequence $\{c_n\} \subset \mathcal{S}$ s.t. for any test curve $b \in \mathcal{S}$ we have

$$\alpha_n \cdot i(b, c_n) \longrightarrow I(\mathcal{F}, b)$$

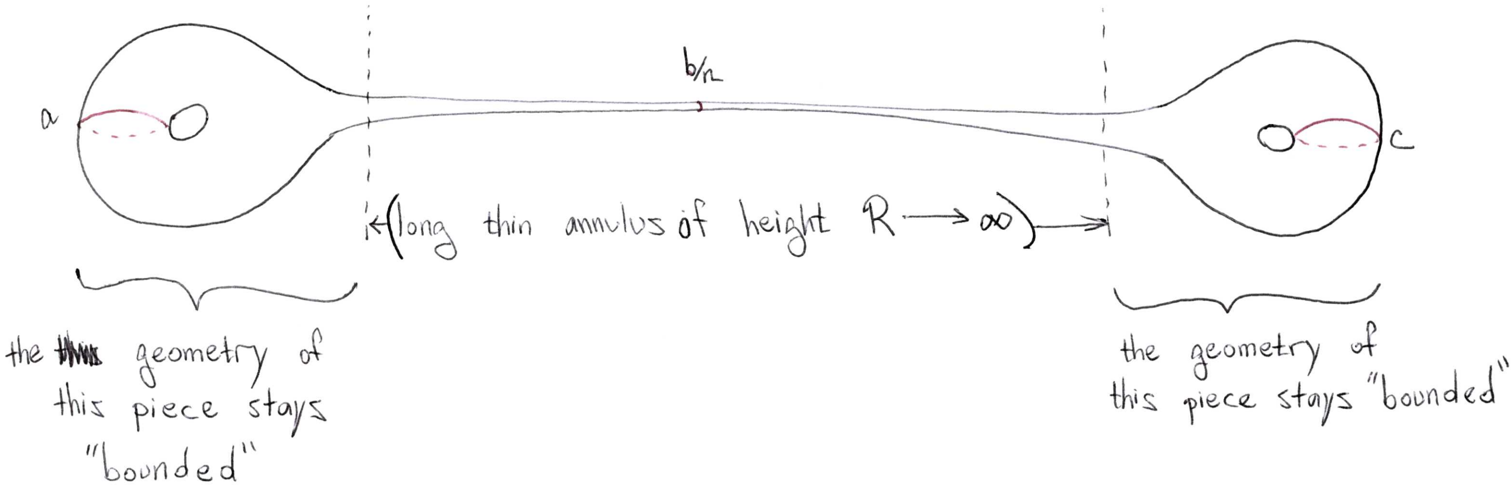
for some sequence of scaling factors α_n.

I won't prove this, at least not now.

How can a sequence of hyperbolic surfaces in $\mathcal{T}$ converge to a simple closed curve in $\mathbb{P}([0,\infty)^{\mathcal{S}})$? Let's give a simple example. Recall ~~our~~ ~~[illegible]~~ the Fenchel-Nielsen coordinates on $\mathcal{T}$. We have ~~$(X(\vec{0},\vec{0}), \mathrm{id}) \in \mathcal{T}$ as a basepoint~~
$(X(a,b,c,\vec{0}), \mathrm{id}) \in \mathcal{T}$ as a basepoint.



Consider the sequence $X_n := X(a, \frac{b}{n}, c, \vec{0})$. (Since we're not twisting, let's ignore ~~basepoints~~ markings, which will all be id.) A little hyperbolic geometry shows that for $n \gg 0$, X_n looks like



Let's call the curve of length b/n β. (Sorry for the bad notation.)

If $\alpha \in \mathcal{S}$ satisfies $i(\alpha,\beta)=0$ then α will stay out of the annulus and $\ell_\alpha(X_n)$ [redacted] remains bounded.

If $i(\alpha,\beta)=k>0$ then [redacted] $k\cdot R$ + constant. $\approx \ell_\alpha(X_n)$.

So consider the sequence $\frac{1}{R}\ell_*(X_n) \subset [0,\infty)^{\mathcal{S}}$.

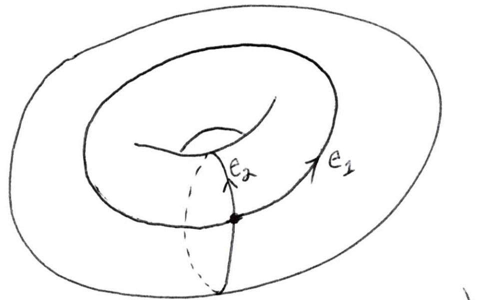
$$\frac{1}{R}\ell_\alpha(X_n) \longrightarrow i(\alpha,\beta) = i_*(\beta).$$

$$\Rightarrow \quad X_n \longrightarrow \beta \quad \text{in} \quad \mathbb{P}([0,\infty)^{\mathcal{S}}).$$

This is the simplest example of a seq. of hyper. surfaces converging to a scc.

The next goal is to show how this works for the humble torus T^2. Since flat tori can be scaled to produce new flat tori, we need a slightly different def'n of $\mathcal{Y}(T^2)$.

$$\mathcal{Y}(T^2) = \left\{ (X,m) \;\middle|\; \begin{array}{l} X \text{ flat } 2\text{-torus with orientable homeo} \\ m: T^2 \to X \text{ s.t. } \ell_{e_1}(X) = 1 \end{array} \right\}$$



[redacted] [redacted]

$$= \left\{ (v_1,v_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \;\middle|\; \begin{array}{l} (v_1,v_2) \text{ positively oriented} \\ \text{and } \quad v_1 = (1,0) \end{array} \right\}$$

↖ clearly this is a dumb condition written this way

(It is convenient to orient e_i of T^2, but as elements of $\mathcal{S}$ they are unoriented.)

$$= \left\{ (x,y) \in \mathbb{R}^2 \mid y > 0 \right\}$$

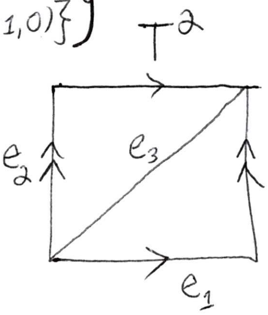
↑ better

The ~~metric~~ bilipschitz metric I defined on $\mathcal{T}(T^2)$ works here, & reproduces the usual Euclidean topology.

What is $\mathcal{S}$?

$$\mathcal{S} = \left\{ \pm(m,n) \in (\mathbb{Z}\times\mathbb{Z})/_{\text{sign}} \;\middle|\; \begin{array}{l} \text{If } m\neq 0\neq n \text{ then} \\ \gcd(m,n)=1, \\ \text{Otherwise } (m,n)\in\{(0,\pm1),(\pm1,0)\} \end{array} \right\}$$

On T^2 define a 3rd scc e_3 as shown.



A curve $c\in\mathcal{S}$ can be written

$$c = \pm(m,n) = \pm(me_1+ne_2)$$

↑ ↗

(To make sense of ~~the~~ this we think of $e_i\in\mathbb{R}^2$)

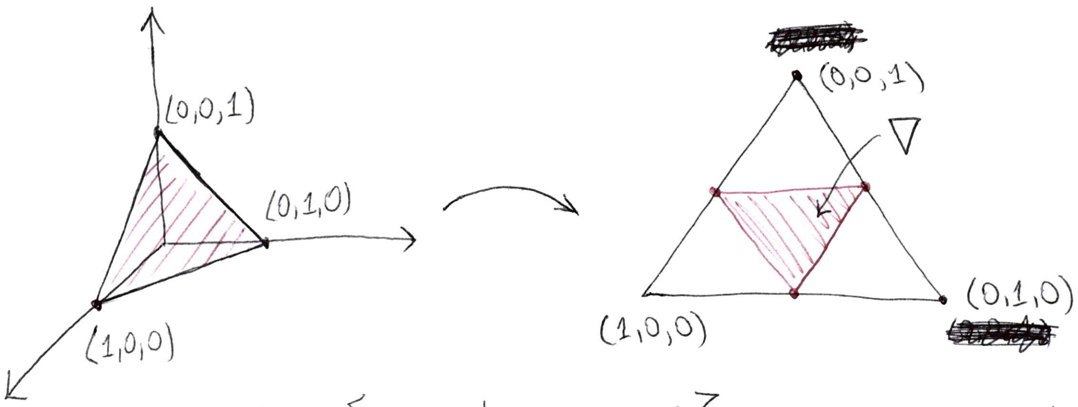
Exercise: $i(c,e_1)=|n|$, $i(c,e_2)=|m|$, $i(c,e_3)=|m-n|$.

Moreover, if $b=\pm(m'e_1+n'e_2)$ then

$$i(b,c) = \left|\det\begin{pmatrix} m & n \\ m' & n'\end{pmatrix}\right|.$$

Notice any pair of numbers from $\{i(c,e_j)\}_{j=1}^{3}$ specifies two elements of $\mathcal{S}$, namely $\pm(m,n)$ and $\pm(m,-n)$. To obtain uniqueness all three numbers are required.

Consider the triangular regular simplex of $\mathbb{R}^3$:



In barycentric coords $\{(x,y,z) \mid x+y+z=1\}$, the region where all 3 triangle inequalities $\left\{\begin{array}{l} x+y\geq z \\ x+z\geq y \\ y+z\geq x\end{array}\right\}$ hold is shown on the right in red. Call it ∇.

Let $\mathrm{Cone}(\nabla) := \{ r\cdot(x,y,z) \mid (x,y,z)\in\nabla \;\&\; r>0 \}$

and $\mathrm{Cone}(\partial\nabla) := \{ r\cdot(x,y,z) \mid (x,y,z)\in\partial\nabla \;\&\; r>0 \}$.

Notice $\mathrm{Cone}(\partial\nabla)$ is the set of points where at least one triangle inequality is an equation.

Define

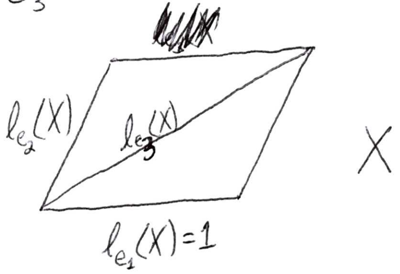
$$\begin{aligned} i_3 : \mathcal{S} &\longrightarrow \mathrm{Cone}(\partial\nabla). \\ c &\longmapsto (i(e_1,c),\, i(e_2,c),\, i(e_3,c)). \end{aligned}$$

i_3 is injective. Also $\mathbb{Q}\cdot\mathrm{image}(i_3) = \mathbb{Q}^3 \cap \mathrm{Cone}(\partial\nabla)$.

So $\mathcal{S}$ forms the rational points of $\mathrm{Cone}(\partial\nabla)$.

Similarly

$$\begin{aligned} l_3 : \mathcal{Y} &\longrightarrow \mathrm{Cone}(\mathrm{int}(\nabla)) \\ X &\longmapsto (\underset{\substack{\uparrow \\ \text{This is always } 1.}}{l_{e_1}(X)},\, l_{e_2}(X),\, l_{e_3}(X)) \end{aligned}$$



By trigonometry, l_3 is injective.

Let $\pi : [0,\infty)^3 \longrightarrow \underset{\uparrow}{\mathbb{P}([0,\infty)^3)} \approx$ simplex be projection.

(Here we're cheating slightly by silently ignoring the origin)

Then $\pi\circ l_3 : \mathcal{Y} \hookrightarrow \mathbb{P}([0,\infty)^3)$ is a homeom. onto the interior of ∇.

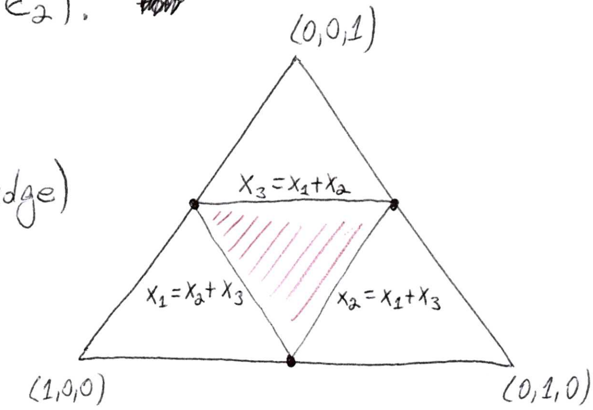
(This is again trigonometry.)

Lemma: For $c \in \mathcal{S}$ $\exists$ map $\varphi_c : \text{cone}(\partial\nabla) \longrightarrow [0,\infty)$ s.t.

- φ_c is continuous
- $\varphi_c(\lambda \vec{x}) = \lambda\, \varphi_c(\vec{x}) \qquad (\lambda > 0)$
- $i(b,c) = \varphi_c(\, i(b,e_1),\ i(b,e_2),\ i(b,e_3))$

Pf sketch: Let $c = \pm(m,n) = \pm(me_1 + ne_2)$.
For $(x,y,z) \in \text{cone}(\partial\nabla)$ define

$$\varphi_c(x,y,z) := \begin{cases} \left|\det\begin{pmatrix} y & -x \\ m & n \end{pmatrix}\right| & \text{if } x_3 = x_1 + x_2 \text{ (top edge)} \\ \left|\det\begin{pmatrix} y & x \\ m & n \end{pmatrix}\right| & \text{otherwise} \end{cases}$$



Let's check a case. If $m,n \geqslant 0$ and $b = \alpha\, e_1 + \beta e_2$ for $\beta \geqslant \alpha \geqslant 0$ then the barycentric coords of β are

$$(i(b,e_1),\ i(b,e_2),\ i(b,e_3)) = (\beta, \alpha, \beta - \alpha).$$ So $x_3 \neq x_1 + x_2$.

$$\varphi_c(\beta, \alpha, \beta-\alpha) = \left|\det\begin{pmatrix} \alpha & \beta \\ m & n \end{pmatrix}\right| = i(b,c). \quad \boxtimes$$

Use φ_c to define $\Phi : \text{cone}(\partial\nabla) \longrightarrow [0,\infty)^{\mathcal{S}}$

$$(x_1,x_2,x_3) \longmapsto \{c \longmapsto \varphi_c(x_1,x_2,x_3)\}.$$

Φ is injective. Recall $i_* : \mathcal{S} \to [0,\infty)^{\mathcal{S}}$.

By construction $i_* = \Phi \circ i_3$.

$$\begin{tikzcd} & \text{cone}(\partial\nabla) \arrow[d, "\Phi"] \\ \mathcal{S} \arrow[ur, "i_3"] \arrow[r, "i_*"'] \arrow[r, phantom, shift left=4, "\circ"] & {[0,\infty)^{\mathcal{S}}} \end{tikzcd}$$

Φ continuous $\Rightarrow$

$(\pi \circ \Phi)(\text{cone}(\partial\nabla)) \subset \overline{\mathcal{S}} \subset \mathbb{P}([0,\infty)^{\mathcal{S}})$.

Claim: The closure of $\mathcal{S}$ in $\mathbb{P}([0,\infty)^{\mathcal{S}})$, i.e. $\overline{(\pi \circ i_*)(\mathcal{S})}$, equals the image $(\pi \circ \Phi)(\text{cone}(\partial \nabla))$.

Pf: Suppose $\{c_n\} \subset \mathcal{S}$ s.t. $(\pi \circ i_*)(c_n)$ converges. Then $\exists$ $\lambda_n > 0$ s.t. $\lambda_n i_*(c_n)$ converges to (nonzero) $g \in [0,\infty)^{\mathcal{S}}$.

$\Rightarrow$ $\forall$ $b \in \mathcal{S}$, $\lambda_n i(c_n, b) \longrightarrow g(b)$.

We want to show $g \in \text{image}(\Phi)$.

$$g(b) = \lim \lambda_n i(c_n, b) = \lim \lambda_n \varphi_b(i(c_n,e_1), i(c_n,e_2), i(c_n,e_3))$$

$$= \varphi_b(g(e_1), g(e_2), g(e_3)) = \Phi(g(e_1), g(e_2), g(e_3)).$$

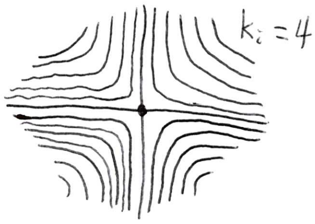
Must check that $(g(e_1), g(e_2), g(e_3)) \in \text{Cone}(\partial \nabla)$, but OK. ☒

This shows $\overline{\mathcal{S}} \approx S^1$ in $\mathbb{P}([0,\infty)^{\mathcal{S}})$.

What about singular measured foliations on T^2?

Let's take a minute to back up and study them.

Thm (Euler - Poincaré Formula): Let M be a closed surface with a measured singular foliation $\mathcal{F}$ and singular set $S = \{p_1, p_2, \dots, p_n\}$. For each p_i let $k_i \in \{3,4,5,\dots\}$ be the number of leaves coming out of p_i. (For example, in the picture $k_i = 4$.) Then



$$2\chi(M) = \sum_{i=1}^{n} (2 - k_i).$$

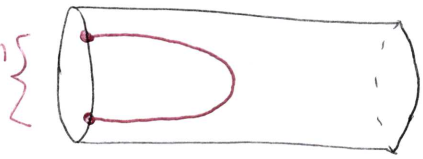
Pf: (See F-L-P, Ch 5. I. 6.).

Cor: If M is a torus then $S = \emptyset$.

Let's examine the torus case. Let $M = T^2$. Fix $\mathcal{F}$ on T^2.

FACT: $\exists$ homotopically nontrivial s.c.c. $c \subset T^2$ transverse to $\mathcal{F}$.
"Pf": Look for a recurrent orbit in the transverse direction. ☒

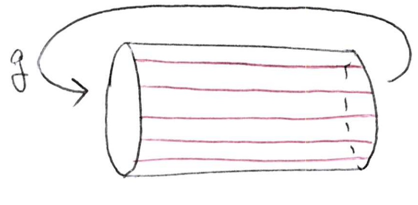
Cut T^2 along c to get an annulus: $\mathcal{F}$ is everywhere transverse to c $\Rightarrow$ a leaf cannot begin and end on the same side of the annulus.



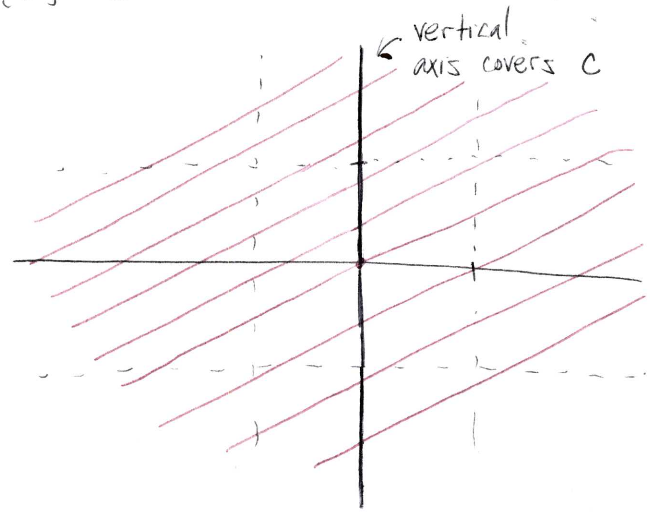
Some pt. in this interval would not be transverse

not possible

So (with some work) one can show the annulus is homeom. to $[0,1] \times S^1$ with leaves $[0,1] \times \{\theta\}$. A gluing homeom. $g: S^1 \to S^1$ recovers T & $\mathcal{F}$. After possibly reparametrizing

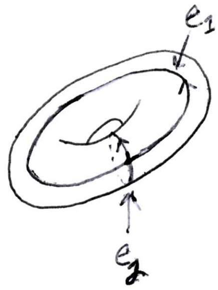


the S^1 factor we can assume g is rotation by some angle θ_0. After gluing by g we obtain coords. $S^1 \times S^1$ for T^2 where $c = \{0\} \times S^1$. In these coords, the universal cover looks like:

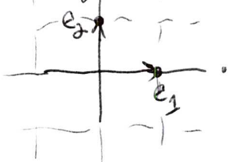


The leaves are straight lines with slope $\theta/2\pi$.

Suppose T^2 can equipped with a marking



and universal cover



. $\exists$ linear map conjugating one universal cover to the other taking c to e_2. So $\mathcal{F}$ will be taken to straight lines in the $\{e_1, e_2\}$ coordinate system also.

Conclusion: $\mathcal{F}$ is isotopic to a foliation on $T^2 = \mathbb{R}^2 / \langle (1,0), (0,1) \rangle$ with straight leaves.

So as a set, and ignoring measure, ~~the~~ foliations on T^2 up to isotopy is just $\mathbb{RP}^1 = \left\{ \begin{smallmatrix} \text{slopes} \\ \text{on } T^2 \end{smallmatrix} \right\}$.

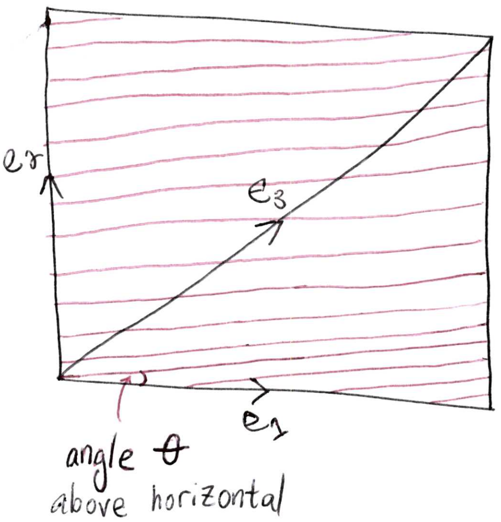
The measure of $\mathcal{F}$ is just determined by the induced measure on the transverse curve c. After possibly another coordinate change we may assume this measure is $|d\theta|$, so it's only invariant is the total mass. Thus, as a set

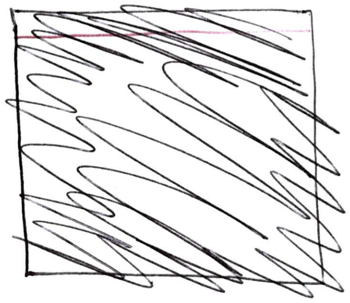
$$\mathcal{MF} = (0,\infty) \times \mathbb{RP}^1 .$$

How did this picture of $\mathcal{MY}$ fit into our previous framework (e.g. i_3, l_3, Φ)? Define

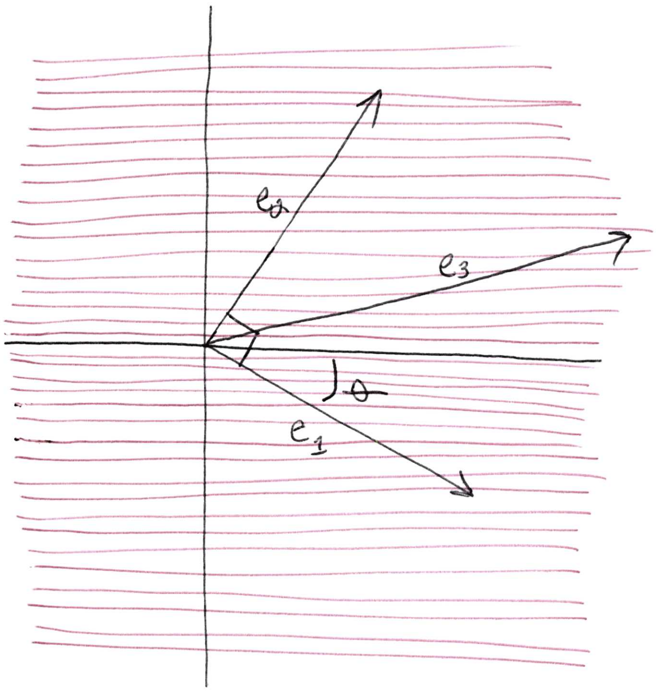
$$I_3 : \mathcal{MY}(T^2) \longrightarrow \mathrm{Cone}(\partial\nabla)$$
$$\mathcal{Y} \longmapsto \big(I(\mathcal{Y},e_1),\ I(\mathcal{Y},e_2),\ I(\mathcal{Y},e_3)\big)$$

where $\mathcal{Y}$ is as shown:





It's easier to compute $I(\mathcal{Y},e_i)$ if we tilt this picture, making $\mathcal{Y}$ horizontal:



Then

$I(\mathcal{Y},e_1) = |\sin\theta|$

$I(\mathcal{Y},e_2) = |\cos\theta|$

$I(\mathcal{Y},e_3) = |\cos\theta - \sin\theta|.$

So I_3 has image in $\mathrm{Cone}(\partial\nabla)$, as claimed.

Recall the topology on $\mathcal{MY}$ was defined by assuming the $I(\mathcal{Y},\cdot)$ fcts are continuous. So I_3 is, in fact, a homeomorphism.

Recall $\Phi : \text{cone}(\partial\nabla) \longrightarrow [0,\infty)^{\mathcal{S}}$, & our thickening construction $\mathcal{S} \xrightarrow{t} \mathcal{M}\mathcal{Y}$.

How do $I_3 \circ t$ & i_3 relate? Fix $c = \pm(m,n) \in \mathcal{S}$ & $t(c) = \mathcal{Y}_c$. Then the angle of $\mathcal{Y}_c$ is $\theta = \arctan\left(\frac{n}{m}\right)$.

So

$$\begin{aligned} I_3(\mathcal{Y}_c) &= (|\sin\theta|, |\cos\theta|, |\cos\theta - \sin\theta|) \\ &= \frac{1}{\sqrt{n^2+m^2}}(|n|, |m|, |m-n|), \end{aligned}$$

& we see $(I_3 \circ t)(c) = \sqrt{n^2+m^2} \cdot i_3(c)$. They're projectively equivalent. This gives the following commutative diagram:

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "i_3"] \arrow[dd, "t"] \arrow[rr, bend left, "i_*", "\circ"'] & \text{Cone}(\partial\nabla) \arrow[r, "\Phi"] \arrow[dd, phantom, "\circ"] & {[0,\infty)^{\mathcal{S}}} \arrow[dr, "\pi"] & \\
 & & & \mathbb{P}([0,\infty)^{\mathcal{S}}) \\
\mathcal{M}\mathcal{Y} \arrow[r, "I_3"] \arrow[rr, bend right, "I_*"', "\circ"] & \text{Cone}(\partial\nabla) \arrow[r, "\Phi"] & {[0,\infty)^{\mathcal{S}}} \arrow[ur, "\pi"'] &
\end{tikzcd}$$