May 27, 2009

Geodesics in Teichmüller space and the curve complex

The next goal is to understand how to build (quasi-) geodesics in the curve complex. Given that the curve complex is not locally finite, it's not obvious how to find them.

To begin we must back up to Teichmüller space. Recall the def'n of Teichmüller space.

$$\mathcal{T}(M) = \left\{ (X,m) \;\middle|\; \begin{matrix} X \text{ hyperbolic surface} \\ \text{w/ homeom. } m: M \to X \end{matrix} \right\} \Big/ \sim$$

where $(X, m_X) \sim (Y, m_Y) \iff \left(\begin{matrix} \exists \text{ isometry } i: X \to Y \\ \text{s.t. } i \circ m_X \sim m_Y \end{matrix} \right).$

Sadly, for this part of the story this is the wrong definition. We need the complex analytic definition

$$\mathcal{T}(M) = \left\{ (X,m) \;\middle|\; \begin{matrix} X \text{ Riemann surface} \\ \text{w/ homeom. } m: M \to X \end{matrix} \right\} \Big/ \sim$$

where $(X, m_X) \sim (Y, m_Y) \iff \left(\begin{matrix} \exists \text{ conformal map } i: X \to Y \\ \text{s.t. } i \circ m_X \sim m_Y \end{matrix} \right).$

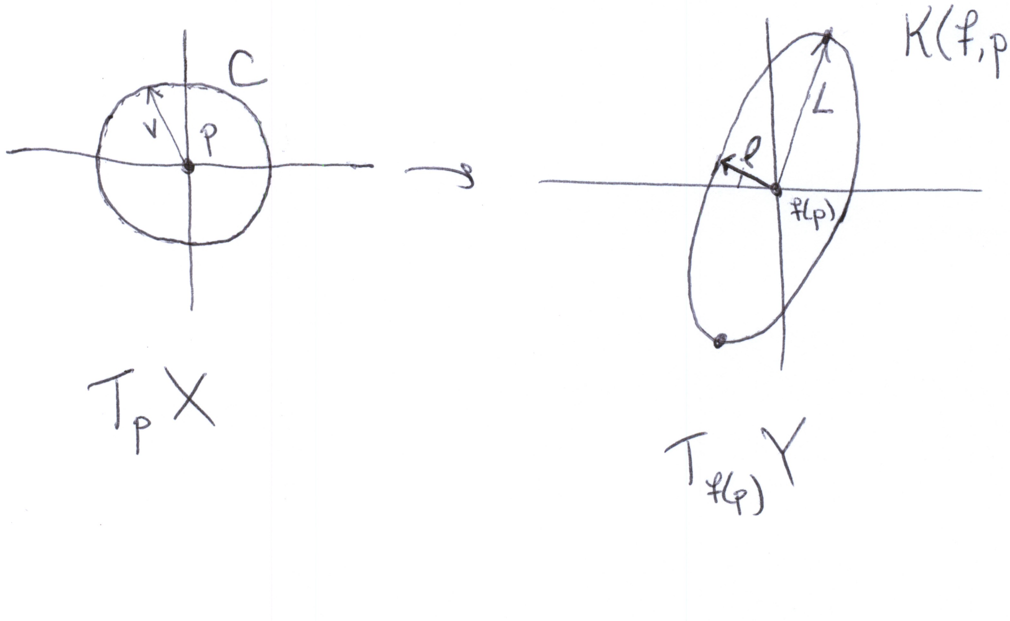
Next we need to know what a quasiconformal map is, at least in the differentiable case.

Sps $f: X \to Y$ is differentiable at $p \in X$. Using only structures on X + Y, we don't know how long vectors are in T_pX + $T_{f(p)}Y$. However, ratios $\frac{\|v_1\|}{\|v_2\|}$ are well defined for v_i in T_pX (or $T_{f(p)}Y$). So pick $v \in T_pX - 0$ and take the circle C of vectors $w \in T_pX$ s.t. $\frac{\|w\|}{\|v\|} = 1$.

Define the quasiconformal constant of f at p to be $\sup_{w_1, w_2 \in C} \frac{\|df_p w_1\|}{\|df_p w_2\|}$, + call it $K(f,p)$. Note $K(f,p) \geq 1$, + $=1$ iff df_p is conformal.

Let the quasiconformal constant of f be the smallest K s.t. $K(f,p) \leq K$ a.e. (We're ignoring many analytic details here, but the a.e. is important. We cannot reasonably require f be differentiable everywhere. That would rule out many or most interesting examples.)

$K(f,p) = L/\ell$.



The Teichmüller metric on $\mathcal{Y}(M)$ is

$$d_{\mathcal{Y}}((X,m_X),(Y,m_Y)) := \frac{1}{2}\log K$$

where $K = \inf \left\{ \begin{matrix} \text{quasiconformal} \\ \text{constant of} \\ \text{homeo. } f \end{matrix} \;\middle|\; \begin{matrix} f: X \to Y \text{ s.t.} \\ m_Y \sim f \circ m_X \end{matrix} \right\}.$

There is a not-so-obvious fact hiding under here: If $K=1$ then $d_{\mathcal{Y}}=0$, so in fact X & Y are conformally equivalent. This means having a q.c. constant of 1 a.e. implies conformality.

Everyone gets to define a metric on $\mathcal{Y}(M)$. Why is this one interesting? Because of Teichmüller's thm., which I'll try to explain next.

Fix $(X,m_X),(Y,m_Y) \in \mathcal{Y}(M)$.

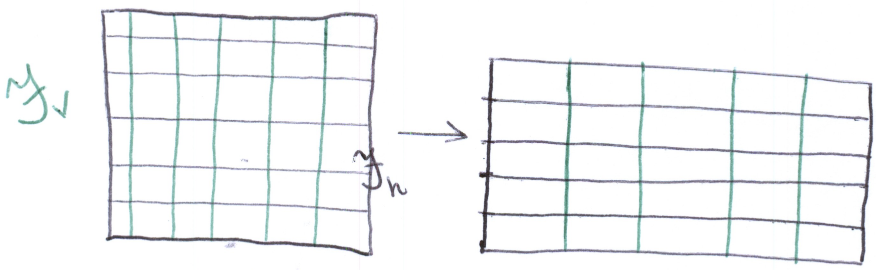
Part 1: $\exists!$ homeom. ~~map~~ $f: X \to Y$ s.t. $m_Y \sim f \circ m_X$ and the q.c. constant of f equals $d_{\mathcal{Y}}((X,m_X),(Y,m_Y))$. In other words the infimum of $d_{\mathcal{Y}}$ is uniquely realized. f is called the Teichmüller map.

Part 2: We can describe the Teichmüller geodesic from (X,m_X) to (Y,m_Y) as (X_t,m_X) for a 1-parameter family of conformal structures X_t, $1 \le t \le e^{2d_{\mathcal{Y}}(X,Y)}$, as follows. Let $X_1 = X$.

Notice this is the qc-constant of f!

$\exists$ a pair of transverse measured singular foliations $\mathcal{F}_h$ and $\mathcal{F}_v$ on X

(think a horizontal foliation + a vertical foliation) such that at a nonsingular pt of X the conformal structure of X_t is given ~~by~~ by stretching by a factor of $\sqrt{t}$ in the horizontal direction and squishing for a factor of $\sqrt{t}$ in the vertical direction.



nonsingular pt.

This defines a path (X_t, m_X) in $\mathcal{Y}(M)$ s.t.

$$(X_{t_0}, m_X) = (Y, m_Y) \text{ for } t_0 = e^{2d_{\mathcal{Y}}(X,Y)}$$

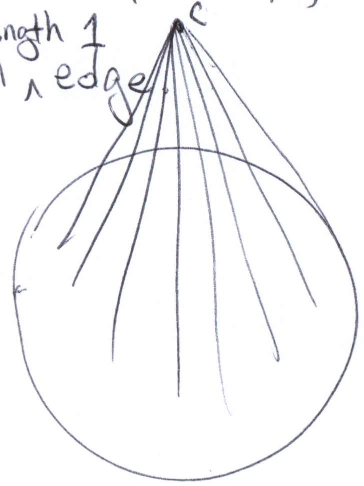
This means the Teichmüller map is <u>affine</u> away from the common singular pts. of $\mathcal{Y}_h$ + $\mathcal{Y}_v$.

Summary: The Teich. metric has unique geodesics with a fairly explicit model for how the surface is changing.

Mention the bi-infinite Teich. geodesic associated to a pseudo-Anosov homeom..

Recall the notion of a Margulis constant μ for hyperbolic surfaces. We need the fact that, on any hyperbolic surface, two ~~simple~~ closed curves of length $< \mu$ never intersect.

Next I'll define the electric Teich. space $\mathcal{T}_{el}(M)$. Start with $\mathcal{T}(M)$ equipped with the Teich. metric. Now for each scc $c \in \mathcal{S}(M)$, add a disjoint point c to $\mathcal{T}(M)$. Finally, if the length of c in (X,m) is less than or $=$ to μ then join (X,m) & c by an length 1 edge.



This makes any pair of surfaces where c is short distance ≤ 2 in $\mathcal{T}_{el}$. There is an obvious map $\mathcal{C} \longrightarrow \mathcal{T}_{el}$ taking a curve c to the added point $c \in \mathcal{T}_{el}$. ~~This map is clearly~~

FACT: For any set of pairwise ~~d~~ disjoint curves $\{c_i\} \subset \mathcal{S}$ on M, $\exists$ $(Y, m_Y) \in \mathcal{T}(M)$ where all the curves c_i are short, i.e. length $\leq \mu$.

Corollary: $\mathcal{C} \longrightarrow \mathcal{T}_{el}$ is 2-Lipschitz.

FACT: ~~Th~~ $\exists$ D s.t. the image of $\mathcal{C} \rightarrow \mathcal{T}_{el}$ is D-dense. I.e. there is always a curve of medium length, and it can be shortened without moving too far in $\mathcal{T}(M)$.

Next define $\Phi : \mathcal{Y} \longrightarrow \left\{ \begin{matrix} \text{subsets of} \\ \mathcal{C} \end{matrix} \right\}$

$$(X, m_X) \longmapsto \left\{ \begin{matrix} \text{the set of shortest} \\ \text{curves on } X \end{matrix} \right\}$$

FACT: $\exists$ $b > 0$ s.t. if $d_{\mathcal{Y}}((X, m_X), (Y, m_Y)) \leq 1$ then

$$\operatorname{diam}(\Phi(X) \cup \Phi(Y)) \leq b.$$

This implies Φ is Lipschitz. We'd like to electrify Φ. This is no problem, just define $\Phi_{el}(c) := c$ for the added points. The resulting map $\Phi_{el} : \mathcal{Y}_{el} \longrightarrow \mathcal{C}$ is Lipschitz, and it is a quasi-inverse to the map $\mathcal{C} \longrightarrow \mathcal{Y}_{el}$.

We conclude that $\mathcal{C} \longrightarrow \mathcal{Y}_{el}$ is a quasi-isometry.

Thm (Masur - Minsky): The map $\Phi : \mathcal{Y} \longrightarrow \mathcal{C}$ sends Teichmüller geodesics to quasi-geodesics of $\mathcal{C}$ with uniform quasigeodesic constants.

This is our desired model of geodesics in $\mathcal{C}(M)$, as images under Φ of Teichmüller geodesics.

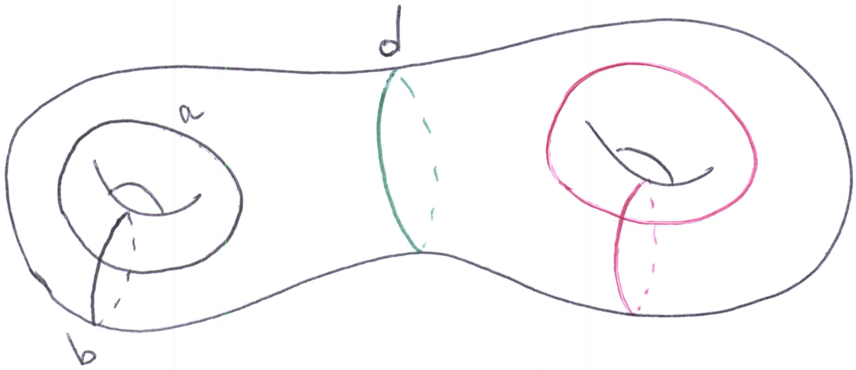
In a related vein, let's finish with a brief introduction to tight geodesics.

Define a geodesic in $\mathcal{C}(M)$ to be a sequence $\Sigma_0, \Sigma_1, \ldots, \Sigma_n$ of simplices of $\mathcal{C}(M)$ s.t.:

$\forall$ i,j and any curves $c_i \in \Sigma_i$, $c_j \in \Sigma_j$,

$d(c_i, c_j) = |i-j|$.

Notice there is a lot of "local" ambiguity in a geodesic of $\mathcal{C}(M)$. For example, let's return to a common example



$d(a,b) = 2$, but if c is any curve in the right half of the surface (3 are shown) then $\{a, c, b\}$ is a geodesic. To eliminate this ambiguity, Masur-Minsky introduced the notion of a tight geodesic. For any pair Σ_i & Σ_{i+2} there is a minimal connected π_1-injective subsurface $R \subset M$ containing all the curves of Σ_i & Σ_{i+2}. As $d(\Sigma_i, \Sigma_{i+2}) = 2$, we know $R \neq M$.

Finally, our geodesic is <u>tight</u> if every curve of Σ_{i+1} is homotopic into ∂R.

For example, in the above picture the only tight geodesic is $\{a, d, b\}$. Notice tightness is local.

Thm (MM): Between any two points of $\mathcal{C}$ $\exists$ a finite number of tight geodesics.