Laminations, train tracks, and singular foliations

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Fix $M$ a closed hyperbolic surface. (To keep things simple today we'll assume $M$ has no boundary.)

Def: A geodesic lamination $\lambda$ on $M$ is a union of disjoint geodesics of $M$ forming a closed set. (Note that a geodesic is either closed or bi-infinite.)

![Diagram of a multicurve]

Each component of $M - \lambda$ has area $\geq \pi$, because it has geodesic boundary. One can apply Gauss-Bonnet.

$\Rightarrow M - \lambda$ has $\leq \frac{\text{Area}(M)}{\pi} = 2 \cdot |\chi(M)|$ components.

With a little more work one can show $\lambda$ has area 0 in $M$. (For this and much more, see Chapter 8 section 5 of Thurston's Notes.)

With a homeomorphism $f: M \to N$, for $N$ a hyperbolic surface, we can push $\lambda$ to $N$ by identifying a geodesic in $M$ with a distinct pair in $\partial M$, and then using the $\pi_1$-equivariant homeomorphism $\partial M \to \partial N$. So the choice of metric on $M$ is just for convenience.
Def: A measured geodesic lamination \( \lambda \) on \( M \) is a geodesic lamination \( \lambda \) together with a measure \( \mu \) on the set of compact arcs of \( M \) transverse to \( \lambda \) satisfying:

1. \( \mu(a) < \infty \) for any compact arc \( a \) transverse to \( \lambda \)

2. If \( \alpha_t \) is a 1-parameter family of compact arcs transverse to \( \lambda \) s.t. \( \alpha_t \cap \lambda = \emptyset \) for all \( t \) then \( \mu(\alpha_0) = \mu(\alpha_t) = t \).

3. We assume \( \mu \) has full support, i.e. \( \alpha_0 \neq \emptyset \) \( \Rightarrow \mu(\alpha) > 0 \).

Note the multicurve with additional "spiral" geodesics from page 1 cannot be made into a measured geodesic lamination. A measured geod. lam' cannot have an infinite geod. spiraling into a closed geod.

One could, in this situation, find a family \( \alpha_t \) of transverse arcs s.t. \( \alpha_1 \neq \alpha_0 \Rightarrow \mu(\alpha_t) = 0 \).
The next goal is to define train tracks. It's best to draw lots of pictures. A branch of track is an embedded square in $M$ with its vertical and horizontal foliation. The horizontal foliation forms the leaves of the branch. The vertical foliation forms the ties.

A switch is a union of glued along boundary ties so there are no dead-end leaves.

A train track $\mathcal{T}$ on $M$ is a collection of branches and switch so there are no dead-end leaves, and no component of $M - \mathcal{T}$ is diffeomorphic to a 1-gon or a bigon

(or a 0-gon)

Note that topology in $M - \mathcal{T}$ is allowed. E.g.
Now assume $A$ is a geodesic lamination with measure $\mu$. Note that inside any fixed branch of $T$ the measure of a tie is constant. Moreover, at a switch the total measure of the ties on the left equals the total measure of the ties on the right. This motivates the def'n

**Def:** A weighted train track assigns a positive weight to each branch such that at each switch the total weights on each side are equal.

So our construction builds a weighted train track from a measured geodesic lamination. Notice that by choosing $\varepsilon$ smaller we obtain finer approximations of our lamination.

Building a measured singular foliation from a weighted train track is easy, if a bit technical to nail down. Simply cover the complement of $T$ and then collapse the rest of the complement of $T$ onto the singular leaves.
Add singular leaves and then collapse.

Keep in mind the ties are transverse to the resulting foliation.

There is ambiguity when choosing how to add singular leaves. All choices are Whitehead equivalent. When a complementary region has some topology then adding singular leaves is slightly more complex. We'll skip these details here. This gives a singular foliation. What about the measure? For each branch of the train track of weight $w$ put a uniform Lebesgue measure on the ties of total measure $w$. This measure transfers in the obvious way to curves transverse to the singular foliation.

This describes maps

\[
\{ \text{measured laminations} \} \rightarrow \{ \text{weighted train tracks} \} \rightarrow \{ \text{measured singular foliations} \}.
\]
We'll complete the picture with a map

\[ \text{measured singular foliations} \rightarrow \text{measured laminations}. \]

Consider a measured singular foliation \( \mathcal{F} \) on \( M \).
Lift \( \mathcal{F} \) to a measured singular foliation \( \mathcal{F} \) on \( \tilde{M} \).
The boundary at infinity \( \partial_\infty M \) is \( S^1 \).
Each smooth leaf of \( \tilde{\mathcal{F}} \) lifts to a curve in \( \tilde{M} \) with endpoints in \( \partial_\infty \tilde{M} \).

Claim: The endpoints of a smooth leaf cannot coincide.

\[ \text{pf:} \quad \text{If so there must be a "dead end" singular leaf, as shown.} \]
\[ \text{This is not allowed. } \square \]

So we can pull each smooth leaf in \( \tilde{M} \) tight to a geodesic with the same endpoints.

FACT: Distinct leaves pull tight to disjoint geodesics.

This defines a \( \pi_1 \)-equivariant map tight: \( \text{smooth leaves} \rightarrow \text{geods in } \tilde{M} \).

The image of tight(\( \mathcal{F} \)) is a \( \pi_1 \)-invariant geodesic lamination \( \lambda \) of \( \tilde{M} \).
For arc \( a \) a transverse to \( \lambda \) define the measure \( \mu(a) \) as the measure of

\[ \text{tight}^{-1}(a \cap \lambda) \].

This defines a \( \pi_1 \)-invariant measured lamination on \( \tilde{M} \) that descends to a measured lamination on \( M \).