Abstract

We begin higher Waldhausen $K$-theory. The main sources for this talk are Chapter 8 of Rognes, Chapter IV.8 of Weibel, and nLab. For the original development, see Friedhelm Waldhausen’s *Algebraic K-theory of spaces* (1985), 318-419.

Remark 1. Let $\mathcal{C}$ be a Waldhausen category. Our goal is to construct the $K$-theory $K(\mathcal{C})$ of $\mathcal{C}$ as a based loop space $\Omega Y$ endowed with a loop completion map $i : |w\mathcal{C}| \to K(\mathcal{C})$ where $w\mathcal{C}$ denotes the subcategory of weak equivalences. This will produce a function $\text{ob} \mathcal{C} \to |w\mathcal{C}| \to \Omega Y$. Further, we’ll require of $K(\mathcal{C})$ certain limit and coherence properties, eventually rendering $K(\mathcal{C})$ the underlying infinite loop space of a spectrum $K(\mathcal{C})$, called the algebraic $K$-theory spectrum of $\mathcal{C}$.

Definition. Let $\mathcal{C}$ be a category equipped with a subcategory $\text{co}(\mathcal{C})$ of morphisms called *cofibrations*. The pair $(\mathcal{C}, \text{co}(\mathcal{C}))$ is a *category with cofibrations* if the following conditions hold.

1. (W0) Every isomorphism in $\mathcal{C}$ is a cofibration.
2. (W1) There is a base point $*$ in $\mathcal{C}$ such that the unique morphism $* \to A$ is a cofibration for any $A \in \text{ob} \mathcal{C}$.
3. (W2) We have a cobase change

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
C & \to & B \cup_A C
\end{array}
\]

Remark 2. We see that $B \bigsqcup C$ always exists as the pushout $B \cup_A C$ and that the cokernel of any $i : A \to B$ exists as $B \cup_A *$ along $A \to *$. We call $A \hookrightarrow B \rightrightarrows B/A$ a *cofiber sequence*.

Definition. A *Waldhausen category* $\mathcal{C}$ is a category with cofibrations together with a subcategory $w\mathcal{C}$ of morphisms called *weak equivalences* such that every isomorphism in $\mathcal{C}$ is a w.e. and the following “Gluing axiom” holds.

1. (W3) For any diagram

\[
\begin{array}{ccc}
C & \xleftarrow{\sim} & A & \to & B \\
\downarrow & & \downarrow & & \downarrow \\
C' & \xleftarrow{\sim} & A' & \to & B'
\end{array}
\]

the induced map $B \cup_A C \to B' \cup_{A'} C'$ is a w.e.

Definition. A Waldhausen category $(\mathcal{C}, w)$ is *saturated* if whenever $fg$ makes sense and is a w.e., then $f$ is a w.e. iff $g$ is.

Definition. We now introduce the main concept to be generalized.

Let $\mathcal{C}$ be a category with cofibrations. Let the *extension category* $S_2 \mathcal{C}$ have as objects the cofiber sequences in $(\mathcal{C}, \text{co}(\mathcal{C}))$ and as morphisms the triples $(f', f, f'')$ such that

\[
\begin{array}{ccc}
X' & \to & X & \to & X'' \\
\downarrow & & f & & \downarrow \\
Y' & \to & Y & \to & Y''
\end{array}
\]

commutes. This is pointed at $* \to * \to *$. 
**Definition.** Suppose an arbitrary triple \((f', f, f'')\) as above has the property that whenever \(f'\) and \(f''\) are w.e., then so is \(f\). Then we say \(\mathcal{C}\) is *extensional* or *closed under extensions*.

**Remark 3.** Say that the morphism \((f', f, f'')\) is a cofibration if \(f'\), \(f''\), and \(Y' \cup_X Y \to Y\) are cofibrations in \(\mathcal{C}\). Say that the same triple is a weak equivalence if \(f'\), \(f\), and \(f''\) are w.e. in \(\mathcal{C}\). This makes \(S_2\mathcal{C}\) into a Waldhausen category.

**Definition.** Let \(q \geq 0\). Let the *arrow category* \(\text{Ar}[q]\) on \([q]\) have as objects ordered pairs \((i, j)\) with \(i \leq j \leq q\) and as morphisms commutative diagrams of the form

\[
\begin{array}{ccc}
  i & \leq & j \\
  \downarrow & & \downarrow \\
  i' & \leq & j'
\end{array}
\]

We view \([q]\) a full subcategory of \(\text{Ar}[q]\) via the embedding \([q] \xrightarrow{k \mapsto (0, k)} \text{Ar}[q]\).

**Remark 4.**

1. Any triple \(i \leq j \leq k\) determines the morphisms \((i, j) \to (i, k)\) and \((i, k) \to (j, k)\). Conversely, any morphism in the arrow category is a composition of such triples.

2. \(\text{Ar}[q] \cong \text{Fun}([1], [q])\) by identifying each pair \((i, j)\) with the functor satisfying \(0 \mapsto i\) and \(1 \mapsto j\).

**Example 1.** The category \(\text{Ar}[2]\) is generated by the commutative diagram

\[
\begin{array}{ccc}
  (0, 0) & \longrightarrow & (0, 1) \\
  \downarrow & & \downarrow \\
  (1, 1) & \longrightarrow & (1, 2) \\
  \downarrow & & \downarrow \\
  (2, 2)
\end{array}
\]

**Definition.** Let \(\mathcal{C}\) be a category with cofibrations and \(q \geq 0\). Define \(S_q\mathcal{C}\) as the full subcategory of \(\text{Fun}(\text{Ar}[q], \mathcal{C})\) generated by \(X : \text{Ar}[q] \to \mathcal{C}\) such that

1. \(X_{j,j} = *\) for each \(j \in [q]\).

2. \(X_{i,j} \to X_{i,k} \to X_{j,k}\) is a cofiber sequence for any \(i < j < k\) in \([q]\). Equivalently, if \(i \leq j \leq k\) in \([q]\), then the square

\[
\begin{array}{ccc}
  X_{i,j} & \longrightarrow & X_{i,k} \\
  \downarrow & & \downarrow \\
  X_{j,j} = * & \longrightarrow & X_{j,k}
\end{array}
\]

is a pushout.

This is pointed at the constant diagram at *.
**Remark 5.** A generic object in $S_q\mathcal{C}$ looks like

\[
\begin{array}{ccccccccc}
* & \rightarrow & X_1 & \leftarrow & \cdots & \rightarrow & X_{q-1} & \rightarrow & X_q \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
* & \rightarrow & X_{q-1}/X_1 & \leftarrow & \cdots & \rightarrow & X_q/X_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* & \rightarrow & X_q/X_{q-1} \\
\end{array}
\]

where $X_q$ corresponds to $X_{0,q}$ and $X_{j}/X_i$ to $X_{i,j}$ for any $1 \leq i \leq j \leq q$.

**Definition.** Let $(\mathcal{C}, co\mathcal{C})$ be a category with cofibrations. Let $coS_q\mathcal{C} \subset S_q\mathcal{C}$ consist of the morphisms $f : X \rightarrow Y$ of $\text{Ar}[q]$-shaped diagrams such that for each $1 \leq j \leq q$ we have

\[
\begin{array}{ccccccccc}
X_{0,j-1} & \rightarrow & X_{0,j} & \leftarrow & \cdots & \rightarrow & X_{0,q-1} & \rightarrow & X_{0,q} \\
Y_{0,j-1} & \rightarrow & X_{0,j} \cup X_{0,j-1} & \leftarrow & \cdots & \rightarrow & X_{0,q} \cup X_{0,q-1} & \rightarrow & X_{0,q} \\
\end{array}
\]

**Proposition 1.** If $f : X \rightarrow Y$ is a cofibration of $S_q\mathcal{C}$, then

\[
\begin{array}{ccccccccc}
X_{i,j} & \rightarrow & X_{i,k} \\
Y_{i,j} & \rightarrow & Y_{i,k} \\
\end{array}
\]

for any $i \leq j \leq k$ in $[q]$.

**Proof.** The proof is mostly an easy induction argument along with an application of Lemma 1 above. See Rognes, Lemma 8.3.12. \hfill \square

**Lemma 1.** $(S_q\mathcal{C}, coS_q\mathcal{C})$ is a category with cofibrations.

**Proof.** First notice that the composite of two cofibrations $g \circ f : X \rightarrow Y \rightarrow Z$ is a cofibration because we have

\[
\begin{array}{ccccccccc}
X_{0,j-1} & \rightarrow & X_{0,j} & \leftarrow & \cdots & \rightarrow & X_{0,q-1} & \rightarrow & X_{0,q} \\
Y_{0,j-1} & \rightarrow & X_{0,j} \cup X_{0,j-1} & \leftarrow & \cdots & \rightarrow & X_{0,q} \cup X_{0,q-1} & \rightarrow & X_{0,q} \\
Z_{0,j-1} & \rightarrow & X_{0,j} \cup X_{0,j-1} & \leftarrow & \cdots & \rightarrow & X_{0,q} \cup X_{0,q-1} & \rightarrow & X_{0,q} \\
\end{array}
\]
It’s clear that any isomorphism or initial morphism in $S_q\mathcal{C}$ is a cofibration.

To see that (W2) is satisfied, let $f : X \to Y$ and $g : X \to Z$ be morphisms in $S_q\mathcal{C}$. It’s easy to verify that each component $f_{i,j} : X_{i,j} \to Y_{i,j}$ is a cofibration. Thus, each pushout $W_{i,j} := Y_{i,j} \cup_{X_{i,j}} Z_{i,j}$ exists. These form a functor $W : \text{Ar}[q] \to \mathcal{C}$. If $i < j < k$, then we have $W_{i,j} \to W_{i,k} \to W_{j,k}$ because the left morphism factors as the composite of two cofibrations

$$
\begin{array}{ccc}
Z_{i,j} & \longrightarrow & Z_{i,k} \\
\downarrow & & \downarrow \\
Y_{i,j} \cup_{X_{i,j}} Z_{i,j} & \longrightarrow & Y_{i,k} \cup_{X_{i,j}} Z_{i,k}
\end{array}
$$

$$
\begin{array}{ccc}
\text{id} \cup g_{i,k} & & \text{id} \cup g_{i,j} \\
\uparrow & & \uparrow \\
Y_{i,j} \cup_{X_{i,j}} X_{i,k} & \longrightarrow & Y_{i,k}
\end{array}
$$

The fact that colimits commute confirms that $W_{j,k} \cong W_{i,k} / W_{i,j}$. Hence $W$ is the pushout of $f$ and $g$. To verify that this is a cofibration, we must check that the pushout map $W_{0,j-1} \cup_{Z_{0,j-1}} Z_{0,j} \to W_{0,j}$ is a cofibration. But this follows from the pushout square

$$
\begin{array}{ccc}
Y_{0,j-1} \cup_{X_{0,j-1}} X_{0,j} & \longrightarrow & Y_{0,j} \\
\downarrow & & \downarrow \\
Y_{0,j-1} \cup_{X_{0,j-1}} Z_{0,j} & \longrightarrow & Y_{0,j} \cup_{X_{0,j}} Z_{0,j}
\end{array}
$$

Definition. Let $(\mathcal{C}, w\mathcal{C})$ be a Waldhausen category. Let $wS_q\mathcal{C} \subseteq S_q\mathcal{C}$ consist of the morphisms $f : X \sim \to Y$ of $\text{Ar}[q]$-shaped diagrams such that the component $f_{0,j} : X_{0,j} \to Y_{0,j}$ is a w.e. in $\mathcal{C}$ for each $1 \leq j \leq q$.

**Proposition 2.** Let $f$ be a w.e. in $S_q\mathcal{C}$. Each component $f_{i,j} : X_{i,j} \to Y_{i,j}$ is a w.e. in $\mathcal{C}$.

**Proof.** Apply the Gluing axiom to the diagram

$$
\begin{array}{ccc}
X_{0,j} & \longrightarrow & X_{0,i} \\
\cong & & \cong \\
Y_{0,j} & \longrightarrow & Y_{0,i}
\end{array}
$$

Then $X_{i,j} \cong X_{0,j} \cup_{X_{0,i}} \ast \sim \to Y_{0,j} \cup_{Y_{0,i}} \ast \cong Y_{i,j}$, as desired.

**Lemma 2.** $(S_q\mathcal{C}, wS_q\mathcal{C})$ is a Waldhausen category.

**Definition.** Let $\mathcal{C}$ be a category with cofibrations. If $\alpha : [p] \to [q]$, then define $\alpha^* : S_q\mathcal{C} \to S_p\mathcal{C}$ by

$$
\alpha^*(X : \text{Ar}[q] \to \mathcal{C}) = X \circ \text{Ar}(\alpha) : \text{Ar}[p] \to \text{Ar}[q] \to \mathcal{C}.
$$

It’s easy to check that this satisfies the two conditions of a diagram in $S_p\mathcal{C}$. Moreover, the face maps $d_i$ are given by deleting the row $X_{i,-}$ and the column containing $X_i$ in $(\ast)$ of Remark 5 and then reindexing as necessary. The degeneracy maps $s_i$ are given by duplicating $X_i$ and then reindexing such that $X_{i+1,i} = 0$. [[Not sure the $s_i$ work.]]

**Proposition 3.** Let $(\mathcal{C}, w\mathcal{C})$ be a Waldhausen category. Each functor $\alpha^* : S_q\mathcal{C} \to S_p\mathcal{C}$ is exact, so that $(S^\bullet \mathcal{C}, wS^\bullet \mathcal{C})$ is a simplicial Waldhausen category.
Remark 6. The nerve $N_* wS_* C$ is a bisimplicial set with $(p,q)$-bisimplices the diagrams of the form

\[ \ast \to X_0^0 \to X_1^0 \to \cdots \to X_q^0 \]
\[ \sim \downarrow \sim \downarrow \sim \downarrow \]
\[ \ast \to X_0^1 \to X_1^1 \to \cdots \to X_q^1 \]
\[ \sim \downarrow \sim \downarrow \sim \downarrow \]
\[ \vdots \vdots \vdots \vdots \]
\[ \sim \downarrow \sim \downarrow \sim \downarrow \]
\[ \ast \to X_0^p \to X_1^p \to \cdots \to X_q^p \]

such that $X_{i,j}^k \cong X_{j,k}^i$ for every $i \leq j \leq q$ and $k \in [p]$.

Lemma 3. There is a natural map $N_* w\mathcal{C} \wedge \Delta^1_* \to N_* wS_* \mathcal{C}$, which automatically induces a based map $\sigma : \Sigma|w\mathcal{C}| \to |wS_* \mathcal{C}|$ of classifying spaces.

Proof. We can treat $N_* wS_* \mathcal{C}$ as the simplicial set $[q] \mapsto N_* wS_q \mathcal{C}$. This defines a right skeletal structure on $N_* wS_* \mathcal{C}$.

If $q = 0$, then $wS_0 \mathcal{C} = S_0 \mathcal{C} = \ast$, so that $N_* wS_0 \mathcal{C} = \ast$ as well. If $q = 1$, then $wS_1 \mathcal{C} \cong w\mathcal{C}$. Thus, the right 1-skeleton is equal to $N_* w\mathcal{C} \wedge \Delta^1_*$, which in turn must be equal to the image $I$ of the canonical map

\[ \prod_{q \leq 1} N_* wS_q \mathcal{C} \times \Delta^1_* \to N_* wS_* \mathcal{C}. \]

Now, the degeneracy map $s_0$ collapses $\{\ast\} \times \Delta^1_*$, and the face maps $d_0$ and $d_1$ collapse $N_* w\mathcal{C} \times \partial \Delta^1_*$. Therefore, $I$ must equal

\[ N_* w\mathcal{C} \wedge \Delta^1_* = \frac{N_* w\mathcal{C} \times \Delta^1_*}{\{\ast\} \times \Delta^1_* \cup N_* w\mathcal{C} \times \partial \Delta^1_*}. \]

We have defined a natural inclusion map $\lambda : N_* w\mathcal{C} \wedge \Delta^1_* \to N_* wS_* \mathcal{C}$.

Since $\Delta^1_*$ is isomorphic to the unit interval and the map $\lambda$ agrees on the endpoints, we can pass to $S^1$ during the suspension. Hence $\lambda$ immediately induces the desired map $\sigma$. [[This is a tentative explanation offered by Thomas.]]

Remark 7. The axiom (W3) implies that $w\mathcal{C}$ is closed under coproducts, making $|wS_* \mathcal{C}|$ into an $H$-space via the map

\[ \prod : |wS_* \mathcal{C}| \times |wS_* \mathcal{C}| \cong |wS_* \mathcal{C} \times wS_* \mathcal{C}| \to |wS_* \mathcal{C}|. \]

Definition. Let $(\mathcal{C}, w\mathcal{C})$ be a Waldhausen category. Define the algebraic $K$-theory space

\[ K(\mathcal{C}, w) = \Omega|N_* wS_* \mathcal{C}|. \]

Then we have a right adjoint $\iota : |w\mathcal{C}| \to K(\mathcal{C}, w)$ to the based map $\sigma$.

Moreover, let $F : (\mathcal{C}, w\mathcal{C}) \to (\mathcal{D}, w\mathcal{D})$ be an exact functor. Then set $K(F) = \Omega|wS_* F| : K(\mathcal{C}, w) \to K(\mathcal{D}, w)$. We have thus defined the algebraic $K$-theory functor $K : Wald \to Top_*$. 

Remark 8. Recall that any exact category $\mathcal{A}$ is a Waldhausen category with cofibrations the admissible exact sequences and w.e. the isomorphisms. Waldhausen showed that $iS_* \mathcal{A}$ (where $i$ denotes the iso category) and $BQ \mathcal{A}$ are homotopy equivalent. Hence our current definition of higher algebraic $K$-theory agrees with Quillen’s.
Example 2. Let \( R \) be a ring. Define the \emph{algebraic K-theory space} of \( R \) as

\[
K(R) = K(P(R), i)
\]

where the w.e. \( i \) are precisely the injective \( R \)-linear maps with projective cokernel and the cofibrations are precisely the \( R \)-linear maps.

Example 3. Assume that \( \mathcal{C} \) is a small Waldhausen category where \( w\mathcal{C} \) consists of the isomorphisms in \( \mathcal{C} \). If \( s_0 \mathcal{C} \) denotes the set of objects of \( S_0 \mathcal{C} \), then we get a simplicial set \( s_\bullet \mathcal{C} \). Waldhausen showed that the inclusion \( |s_\bullet \mathcal{C}| \to |S_\bullet \mathcal{C}| \) is a homotopy equivalence. This makes \( \Omega |s_\bullet \mathcal{C}| \) into a so-called simplicial model for \( K(\mathcal{C}, w) \).

Remark 9. Since \( wS_0 \mathcal{C} = \ast \) and every simplex of degree \( n > 0 \) is attached to \( \ast \), it follows that the classifying space \( |wS_\bullet \mathcal{C}| \) is connected. Therefore, we preserve any homotopical information when passing to the loop space.

Definition. Define the \( i \)-th \emph{algebraic K-group} as \( K_i(\mathcal{C}, w) = \pi_i K(\mathcal{C}, w) \) for each \( i \geq 0 \).

Proposition 4. \( \pi_1 |wS_\bullet \mathcal{C}| \cong K_0(\mathcal{C}, w) \).

Lemma 4. The group \( K_0(\mathcal{C}, w) \) is generated by \( [X] \) for every \( X \in \text{ob} \mathcal{C} \) such that \( [X'] + [X''] = [X] \) for every cofiber sequence \( X' \to X \to X'' \) and \( [X] = [Y] \) for every \( X \sim Y \).

Proof. We compute \( \pi_1 |N_\bullet wS_\bullet \mathcal{C}| \) based at the \( (0, 0) \)-bisimplex \( \ast \). Notice that \( |N_\bullet wS_\bullet \mathcal{C}| \) has a CW structure [this is reasonable visually] with 1-cells the \( (0, 1) \)-bisimplices and 2-cells the \( (0, 2) \)-bisimplices \( X' \to X \to X'' \) and the \( (1, 1) \)-bisimplices \( X \sim Y \), which are attached to the 1-cells \( X \) and \( Y \). Any cell of dimension \( n > 2 \) is irrelevant to computing \( \pi_1 \).

Corollary 1. We obtain the functors \( K_i : \text{Wald} \to \text{Top}, \to \text{Ab} \), called the \emph{algebraic K-group functors}.

Proof. By Proposition 4, we know that \( K_i(\mathcal{C}, w) = \pi_{i+1} |wS_\bullet \mathcal{C}| \), which is abelian for \( i \geq 1 \). Moreover, note that if \( X' \to X' \vee X'' \to X'' \) and \( X'' \to X' \vee X'' \to X' \) are cofiber sequences, then the previous lemma implies that \( [X'] + [X''] = [X' \vee X''] = [X'' + X'] \). Hence \( K_0(\mathcal{C}, w) \) is also abelian.

Example 4. Let \( X \) be a CW complex and \( \mathcal{R}(X) \) denote the category of CW complexes \( Y \) obtained from \( X \) by attaching at least one cell such that \( X \) is a retract of \( Y \). Equip this with cofibrations in the form of cellular inclusions fixing \( X \) and \( w.e. \) in the form of homotopy equivalences. This makes \( \mathcal{R}(X) \) into a Waldhausen category. If \( \mathcal{R}_f(X) \) denotes the subcategory of those \( Y \) obtained by attaching finitely many cells, then we write \( A(X) := K(\mathcal{R}_f(X)) \).

Lemma 5. \( A_0(X) \cong \mathbb{Z} \).

Proof. Weibel leaves this proof as an exercise.

Definition. If \( \mathcal{B} \) is a Waldhausen subcategory of \( \mathcal{C} \), then it is \emph{cofinal in} \( \mathcal{C} \) is for any \( X \in \text{ob} \mathcal{C} \), there is some \( X' \in \text{ob} \mathcal{C} \) such that \( X \amalg X' \in \text{ob} \mathcal{B} \).

Theorem 1. Let \( (\mathcal{B}, w) \) be cofinal in \( (\mathcal{C}, w) \) and closed under extensions. Assume that \( K_0(\mathcal{B}) = K_0(\mathcal{C}) \). Then \( wS_\bullet \mathcal{B} \to wS_\bullet \mathcal{C} \) is a homotopy equivalence. Therefore, \( K_i(\mathcal{B}) \cong K_i(\mathcal{C}) \) for every \( i \geq 0 \).