Simplicial Sets

Simplicial sets are model for topological spaces

Recall Δ skeleton category w/ objects

$$[n] = \{0 < 1 < \cdots < n\}. \qquad n \in \mathbb{Z}^+$$

$\Delta([m],[n])$ are set of order preserving functions.

$$\alpha : [m] \to [n] \quad \text{s.t.}$$

i.e. $\quad i \le j \Rightarrow \alpha(i) \le \alpha(j)$

For $n \ge 1$, the i-th coface map $\quad \delta_i = \delta_i^n : [n-1] \to [n]$

is given by

$$\delta_i(j) = \begin{cases} j & \text{for } j < i \\ j+1 & \text{for } j \ge i \end{cases} \qquad n \ge 1, \quad 0 \le i \le n.$$

For $n \ge 0$, $0 \le j \le n$, the j-th degeneracy map

$$\sigma_j = \sigma_j^n : [n+1] \to [n]$$

is given by

$$\sigma_j(i) = \begin{cases} i & \text{for } i \le j \\ i-1 & \text{for } i > j \end{cases}$$

~~co~~ simplicial identities.

$$\delta_j \delta_i = \delta_i \delta_{j-1} \qquad i < j$$

$$\sigma_j \delta_i = \delta_i \sigma_{j-1} \qquad i < j$$

$$\sigma_j \delta_i = id \qquad j \le i \le j+1$$

$$\sigma_j \delta_i = \delta_{i-1} \sigma_j \qquad j+1 < i$$

$$\sigma_j \sigma_i = \sigma_i \sigma_{j+1} \qquad i \le j$$

Any morphism ~~can~~ $\alpha : [m] \to [n]$ can be factored as

$$\alpha = \delta_{i_r} \cdots \delta_{i_1} \sigma_{j_1} \cdots \sigma_{j_s}.$$

simplicial set $X_\cdot$ is a contravariant functor

$X_\cdot : \Delta^{op} \to \underline{Set}$. $X_n = X_\cdot([n])$

$\alpha : [m] \to [n] \to \alpha^* = X(\alpha) : X_n \to X_m$.

i-th face map : $d_i = \delta_i^* : X_n \to X_{n-1}$

j-th degeneracy map $s_j = \sigma_j^* : X_n \to X_{n+1}$

$\Rightarrow$ Simplicial identities.

In fact, a sequence of sets X_n ~~to ge~~ together with

$d_i : X_n \to X_{n-1}$, $0 \le i \le n \ge 1$, $s_j : X_n \to X_{n+1}$

s.t. satisfies simplicial identities specify a

simplicial set $X_\cdot$ uniquely.

<u>sSet</u> : category of simplicial sets.

simplical map : $f_\cdot : X_\cdot \to Y_\cdot$ is a natural

transformation $f : X \Rightarrow Y$.

equivalently, we can specify a simplicial map

$f_\cdot : X_\cdot \to Y_\cdot$ degreewise, i.e.

~~f_n f_m~~ $f_n : X_n \to Y_n$

s.t. $\forall \ \alpha : [m] \to [n]$

$$\begin{array}{ccc} X_n & \to & Y_n \\ \downarrow \alpha^* & & \downarrow \alpha^* \\ X_m & \to & Y_m \end{array}$$

Δ^n standard n-simplex. $\Delta^n = \{(t_0, \cdots, t_n) \mid \sum_{i=1}^{n} t_i = 1, \ t_i \ge 0\}$

$\alpha : [m] \to [n]$ order preserving, we have.

$\alpha^* : \Delta^m \to \Delta^n$ s.t. $\alpha_*(e_j) = e_{\alpha(j)}$

so. $\alpha_* \left(\sum_{j=0}^{m} u_j e_j \right) = \sum_{j=0}^{m} u_j e_{\alpha(j)}$

for $(u_0, \cdots, u_m) \in \Delta^m$

equivalently, $\alpha_*(u_0, u_1, \cdots, u_m) = (t_0, \cdots, t_n)$

with $t_i = \sum_{\alpha(j)=i} u_j$

i-th face map. $0 \le i \le n \ge 1$

$\delta_{i*} : \Delta^{n-1} \to \Delta^n$ is the embedding onto i-th face of Δ^n.

$\delta_{i*}(u_0, \cdots, u_{n-1}) = (u_0, \cdots, u_{i-1}, 0, u_{i+1}, \cdots, u_{n-1})$

boundary of $\Delta^n = \partial \Delta^n = \bigcup_{i=1}^{n} \delta_{i*}(\Delta^{n-1})$

$\sigma_{j*} : \Delta^{n+1} \to \Delta^n$ $\qquad 0 \le j \le n$.

$\sigma_{j*}(u_0, \cdots, u_{n+1}) = (u_0, \cdots, u_{j-1}, u_j + u_{j+1}, u_{j+2}, \cdots, u_{n+1})$

which collapses edge between e_j and e_{j+1}

Now we have a covariant functor $\Delta^{(-)} : \Delta \to \underline{Top}$

$[n] \longrightarrow \Delta^n$

$\alpha \longmapsto \alpha_*$

<u>Geometric realization</u> $\quad X.$ a simplicial set.

$$|X.| = \coprod_{n \ge 0} X_n \times \Delta^n / \sim$$

where $\sim$ is the equivalence relation generated by

$$(x, \alpha_*(\xi)) \sim (\alpha^*(x), \xi).$$

for all $\alpha : [m] \to [n]$ in Δ, $x \in X_n$, $\xi \in \Delta^m$

In particular, this is generated by

$$(x, \delta_{i*}(\xi)) \sim (d_i(x), \xi)$$

<u>Ex.</u> $x \in X_n$ is an "abstract simplex"

this gives rise to a Euclidean n-simplex $\{x\} \times \Delta^n$.

Let y be a point in $\partial\Delta^n$, so $y = \delta_{i*}(\xi)$ where $\xi \in \Delta^{n-1}$, then $(x, \delta_{i*}(\xi)) \sim (d_i(x), \xi)$ ~~tells~~ which is a point in $(n-1)$ simplex $\{d_i(x)\} \times \Delta^{n-1}$

On the other hand, consider n-simplices of the form $s_j(x)$, $0 \le j \le n$, $x \in X_{n-1}$. The corresponding Euclidean n-simplex $\{s_j(x)\} \times \Delta^n$ is identified as $x \times \Delta^{n-1}$ where ~~we~~ $\sigma_{j*} : \Delta^n \to \Delta^{n-1}$ collapsing the edge between e_j and e_{j+1}. Hence it doesn't contribute to new points in $|X_\cdot|$.

Let Y be any topological space. The singular simplicial set $\text{sing}(Y)_\cdot$ is a simplicial set with n-simplices

$$\text{sing}(Y)_n = \underline{\text{Top}}(\Delta^n, Y).$$

the set of maps $\sigma : \Delta^n \to Y$.

For $\alpha : [m] \to [n]$, $\alpha^*(\sigma) = \sigma \circ \alpha_* : \Delta^m \to Y$.

In particular, $d_i(\sigma) = \sigma \circ \delta_{i*}$

$\exists$ bijection $\underline{\text{Top}}(|X_\cdot|, Y) \cong \underline{\text{sSet}}(X_\cdot, \text{sing}(Y)_\cdot)$

i.e.

$$\underline{\text{sSet}} \underset{\text{sing}(-)_\cdot}{\overset{|-|}{\rightleftarrows}} \underline{\text{Top}}.$$

Let the simplicial n-simplex $\Delta^n_\cdot$ be the functor

$\Delta(-,[n])$ with p-simplices $\Delta^n_p = \Delta([p],[n])$
with structure maps $\beta^*: \Delta^n_p \to \Delta^n_q$ for
each $\beta: [q] \to [p]$. (Note: this comes from Yoneda embedding).

For each map $\alpha: [m] \to [n]$, we have a simplicial map
$\alpha_\bullet : \Delta^m_\bullet \to \Delta^n_\bullet$

__Lemma__: $\exists$ a natural bijective correspondance
$\underline{sSet}(\Delta^n_\bullet, X_\bullet) \cong X_n$
taking a simplicial map $f: \Delta^n_\bullet \to X_\bullet$ to
$f_n(id_{[n]}) \in X_n$. The inverse takes
an n-simplex $x \in X_n$ to the characteristic map
$x_\bullet : \Delta^n_\bullet \to X_\bullet$ s.t. in degree p,
$x_p(\zeta) = \zeta^*(x)$ for $\zeta: [p] \to [n]$ in Δ^n_p.

__Lemma__: The rules $[n] \mapsto \Delta^n_\bullet$, $\alpha \mapsto \alpha_\bullet$ defines
a (covariant functor)
$\Delta_\bullet^{(-)} : \Delta \to \underline{Set}$

__Lemma__. $\exists$ a natural homeomorphism $|\Delta^{(-)}_\bullet| \cong \Delta^{(-)}$
of functors $\Delta \to Top$.

__Note__: Consider the map $\coprod_{p \geq 0} \Delta^n_p \times \Delta^p \to \Delta^n$

i.e. $\exists$ homeomorphism $|\Delta^n_\bullet| \cong \Delta^n$ and
$\forall n \geq 0$, $|\alpha_\bullet| : |\Delta^m_\bullet| \to |\Delta^n_\bullet|$ corresponds to
$\alpha_*: \Delta^m \to \Delta^n$ for $\alpha: [m] \to [n]$.

Bisimplicial sets

Def: A bisimplicial set $X_{\bullet,\bullet}$ is a contravariant functor $X: \Delta^{op} \times \Delta^{op} \to \underline{Set}$.
Write $X_{m,n} = X([m],[n])$

Lemma. The category $\underline{ssSet}$ of bisimplicial set is identified with the category $s\underline{sSet}$ of simplicial objects in simplicial sets, via.
$\underline{ssSet} = Fun(\Delta^{op} \times \Delta^{op}, \underline{Set}) \cong Fun(\Delta^{op}, Fun(\Delta^{op}, \underline{Set}))$
$= s\underline{sSet}$

A bisimplicial set $X_{\bullet,\bullet}$ then correspondes to the simplicial simplicial set $[m] \mapsto X_{m,\bullet}$ with the m-simplices the simplicial set $X_{m,\bullet} : [n] \mapsto X_{m,n}$ with simplicial structure map $\beta^* = (id_{[m]}, \beta)^*$.
$[m] \mapsto X_{m,\bullet}$ has simplicial structure map $\alpha^*_\bullet$ with n-th component $\alpha^*_n = (\alpha, id_{[n]})^*$.

Remark: in Waldhausen K-theory, we shall construct ~~bisim~~ geometric realization of a bisimplicial set associated to a simplicial category.

degreewise geometric realization of $X_{\bullet,\bullet}$ is the simplicial space

$$[m] \mapsto |X_{m,\bullet}| = \coprod_{n \geq 0} X_{m,n} \times \Delta^n / \sim,$$

given by the composite $\Delta^{op} \xrightarrow{X} \underline{sSet} \xrightarrow{|\cdot|} \underline{Top}$

<u>Def</u>: the total ~~topo~~ geometric realization of $X_{\bullet,\bullet}$ is the identification space

$$\| X_{\bullet,\bullet} \| = \coprod_{m,n\geq 0} X_{m,n} \times \Delta^m \times \Delta^n / \approx$$

where $\approx$ is generated by

$$(x, (\alpha,\beta)_*(\xi,\eta)) \approx ((\alpha,\beta)^*(x), (\xi,\eta))$$

for $\alpha : [p] \to [m]$, $\beta : [q] \to [n]$, $x \in X_{m,n}$, $\xi \in \Delta^p$, $\eta \in \Delta^q$.

<u>Lemma</u>: $\exists$ a natural homeomorphism $\| X_{\bullet,\bullet} \| \cong \big| [m] \mapsto |X_{m,\bullet}| \big|$

<u>Def</u>: A map $f_{\bullet,\bullet} : X_{\bullet,\bullet} \to Y_{\bullet,\bullet}$ of bisimplicial sets is a weak homotopy equivalence if the total realization $\| f_{\bullet,\bullet} \| : \| X_{\bullet,\bullet} \| \longrightarrow \| Y_{\bullet,\bullet} \|$ is a homotopy equivalence.

<u>Def</u> The simplicial realization of a bisimplicial set $X_{\bullet,\bullet}$ is the simplicial set

$$\coprod_{m\geq 0} X_{m,\bullet} \times \Delta^m_\bullet / \sim,$$

where $(x, \alpha_\bullet(\zeta)) \sim (\alpha^*_\bullet(x), \zeta)$ for $\alpha : [p] \to [m]$, $x \in X_{m,n}$, $\zeta \in \Delta^p_n$. ~~com~~ This comes from the left hand (external) simplicial structure on $X_{\bullet,\bullet}$.

<u>Lemma</u> The geometrical realization of simplicial realization is naturally homeomorphic to the total realization.

Def: The diagonal of a bisimplicial set $X_{\bullet,\bullet}$ is the simplicial set $diag(X)_\bullet$, with n-simplices

$$diag(X)_n = X_{n,n}.$$

Note: any map $f_{\bullet,\bullet} : X_{\bullet,\bullet} \to Y_{\bullet,\bullet}$ induces $diag(f)_\bullet :$ $diag(X)_\bullet \to diag(Y)_\bullet$

Hence $diag : \underline{ssSet} \to \underline{sSet}$ is a functor.

Prop. The diagonal of $X_{\bullet,\bullet}$ is naturally isomorphic to its simplicial realization, hence

$$|diag(X)_\bullet| \cong \| X_{\bullet,\bullet} \|.$$

i.e.

$$\coprod_{p \geq 0} X_{p,p} \times \Delta^p / \sim \quad \cong \coprod_{m,n \geq 0} X_{m,n} \times \Delta^m \times \Delta^n / \approx$$

Note this generalizes to multi-simplicial set.

Hence $f_{\bullet,\bullet} : X_{\bullet,\bullet} \to Y_{\bullet,\bullet}$ is a w.h.e. iff $diag(f)_\bullet : diag(X)_\bullet \to diag(Y)_\bullet$ is.

Prop. (Realization lemma)

Let $f_{\bullet,\bullet} : X_{\bullet,\bullet} \to Y_{\bullet,\bullet}$ be a bisimplicial map, s.t. $\forall m \geq 0$ $f_{m,\bullet} : X_{m,\bullet} \to Y_{m,\bullet}$ is a w.h.e. The $f_{\bullet,\bullet}$ is a w.h.e.

Simplicial homotopy.

DEF. Let ~~$f., g.$~~ $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$ be simplicial maps, a simplicial homotopy is a simplicial map

$$H_\bullet : X_\bullet \times \Delta^1_\bullet \to Y_\bullet$$

s.t. $\quad H_\bullet \circ \delta_{1\bullet} = f_\bullet \quad , \; H_\bullet \circ \delta_{0\bullet} = g_\bullet$

Note: In general this gives no equivalence relation.

A simplicial homotopy $H_\bullet : f_\bullet \simeq g_\bullet$ of simplicial maps induces a homotopy $|H_\bullet| : |f_\bullet| \simeq |g_\bullet|$ of maps $|X_\bullet| \to |Y_\bullet|$

Pf: $\exists$ a ~~comp~~ composite.

$$|X_\bullet| \times |\Delta^1_\bullet| \cong |X_\bullet \times \Delta^1_\bullet| \xrightarrow{|H_\bullet|} |Y_\bullet|$$

since $|\Delta^1_\bullet| = \Delta^1 \simeq I$, $\quad |\delta_{1\bullet}|$ and $|\delta_{0\bullet}|$ corresponds to end points inclusion i_1 and i_0. $\square$.

Homotopy category

Let $Ho(\underline{Top})$, ~~be~~ the homotopy category of $\underline{Top}$, denote the category of topological spaces s.t. all weak equivalences are localized.

Note: Any topological spaces are weakly equivalent to CW complexes. Hence $Ho(\underline{Top})$ is equivalent to category of CW complexes with morphisms are continuous map mod homotopy equivalence. (Note that w.h.e. in CW complexes are h.e.).

THM (Quillen) The adjunction induces an equivalence.

$$|-| : Ho(\underline{sSet}) \rightleftarrows Ho(\underline{Top}) : Sing$$

In particular, this provides a Quillen equivalence between the standard model category structures on these categories.

Among simplicial sets, those that behaves like topological space are ~~called~~ known as Kan complexes or fibrant complexes.

Def: The horn for the k-th face of the n-simplex, denoted by $\Lambda_k[n]$, is the sub simplicial set of $\Delta[n]$ by the union of all face $\Delta[n-1] \hookrightarrow \Delta[n]$, except for the k-th. More generally, consider Δ^n.

Def: We call a simplicial set X a Kan complex if for every map of a horn $\Lambda_k[n]$ to X,

We can extend it to a map $\Delta[n] \to X$.
i.e. we can fill in the dotted arrow

$$\begin{tikzcd} \Lambda_k[n] \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

It is called an n-groupoid if the extension is unique for all $k > n$ and all $i = 0, \dots, k$.

In general, one can also find a "fibrant replacement" of $X_\bullet$. ~~(i.e.)~~, e.g. take $\mathrm{Sing}\,|X_\bullet|$ or via Kan's Ex^∞ functor, that is weakly equivalent to a Kan complex.

Simplicial set and homological algebra

Def: A simplicial abelian group is a simplicial object $A_\bullet$ in $\underline{Ab}$, i.e. a functor $A: \Delta^{op} \to \underline{Ab}$

Note we could define simplicial vector space / R-mod etc.

Define the unnormalized chains $CA_\bullet$ to be the cochain complex

$$(CA_\bullet)^m = \begin{cases} A_{|m|} & m \le 0 \\ 0 & m > 0. \end{cases}$$

with differential $d: (CA_\bullet)^m \longrightarrow (CA_\bullet)^{m+1}$

$$a \longmapsto \sum_{k=0}^{|m|} (-1)^k A_\bullet(\delta_k)(a)$$

where δ_k run over all coface map from $[|m|-1]$ to $[|m|]$

The normalized chains is the cochain complex.

$$(NA_\bullet)^m = \bigcap_{k=0}^{|m|-1} \ker A_\bullet(\delta_k).$$

where $m \le 0$ and δ_k run over all coface map from $[|m|-1]$ to $[|m|]$. The differential is $A(\delta_{|m|})$.

Check: $NA_\bullet \hookrightarrow CA_\bullet$ is a quasi-isomorphism (in fact, a cochain homotopy equivalence).

Ex. Given a topological space X, we construct a simplicial set $\mathrm{Sing}\, X$, then we apply free abelian group functor $\mathbb{Z}(-) : \mathrm{Sets} \to \underline{Ab}$. Now we get a simplicial abelian group

$$\mathbb{Z}\,\mathrm{Sing}\, X : [n] \mapsto \mathbb{Z}(\underline{\mathrm{Top}}(\Delta^n, X))$$

Then we apply the unnormalized chain functor to obtain a singular chain complex

$$C_*(X) = C\,\mathbb{Z}\,\mathrm{Sing}(X).$$

We obtain cochain complexes from simplicial groups. The Dold-Kan tells us we are free to work on either objects.

Let $Ch^{\le 0}(\underline{Ab})$ denote the category of cochain complexes concentrated in non-positive degrees, and let $s\underline{Ab}$ be the category of simplicial abelian groups.

THM (Dold-Kan correspondance) The normalized chain functor $N: sAb \to Ch^{\leq 0}(Ab)$ is an equivalence of categories. Under this correspondance, for all $n \geq 0$,

$$\pi_n(A_\bullet) \cong H^{-n}(NA_\bullet)$$

and all simplicial homotopies go to chain homotopies

In fact, this generalizes to abelian categories.

Simplicial homotopy groups

Given a Kan complex $X_\bullet$ and a base point $* \in X_0$.

Set $Z_n = \{x \in X_n : d_i(x) = * \ \forall i = 0, \ldots, n\}$, here we write $*$ for $s_0^n(*)$.

We say two elements x, x' of Z_n are homotopic, i.e. $x \sim x'$, if $\exists\, y \in X_{n+1}$ s.t.

$$d_i(y) = \begin{cases} * & i < n \\ x & i = n \\ x' & i = n+1 \end{cases}$$

Lemma. If X is a Kan complex, then $\sim$ is an equivalence relation. We define $\pi_n(X) = Z_n/\sim$.

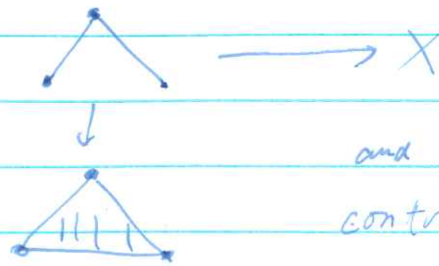
Note: $\pi_n(X)$ agrees with $\pi_n|X|$

If X is not a Kan complex, we define $\pi_n(X) = \pi_n(\operatorname{Sing} X)$

Kan condition

Examples Sing X is a Kan complex.

Let $n=2$. The diagram looks like



and the filler is given by contracting

first before mapping to X.

Ex. (Δ^1) Δ^1 is not a Kan complex.

Consider $\Lambda^2_0 =$