Further applications of Bessel's functions

1. Vibrations of a circularly symmetric membrane

Consider the vibrations of a circular membrane

$$u_{tt} = c^2 \left(\frac{1}{r}[ru_r]_r + \frac{1}{r^2} u_{\theta\theta} \right), \qquad 0 < r < a, \quad -\pi < \theta < \pi \tag{1}$$

with zero boundary conditions

$$u(a, \theta) = 0, \qquad -\pi < \theta \le \pi \tag{2}$$

and radially symmetric initial conditions

$$u(r, \theta, 0) = u_0(r); \qquad u_t(r, \theta, 0) = v_0(r) \tag{3}$$

The solution to the problem (1-3) is then radially symmetric, $u(r, \theta, t) = u(r, t)$ such that we are looking for product solutions

$$u(r, t) = h(t)\phi(r)$$

Replacing in (1) it results,

$$\frac{1}{c^2} \frac{h''}{h} = \frac{(r\phi')'}{r\phi} = -\lambda$$

for some constant $\lambda > 0$ (why?) such that we have

$$h'' + \lambda c^2 h = 0 \Rightarrow h(t) = A \cos(c\sqrt{\lambda} t) + B \sin(c\sqrt{\lambda} t)$$

with A, B arbitrary constants, and

$$\begin{aligned} (r\phi')' + \lambda r\phi &= 0 & (4) \\ \phi(a) &= 0 & (5) \\ |\phi(0)| &< \infty & (6) \end{aligned}$$

Using the change of variable

$$z = \sqrt{\lambda} r, \quad \Phi(z) = \phi(r)$$

the corresponding problem for $\Phi(z)$ is written

$$\begin{aligned} z^2\Phi'' + z\Phi' + z^2\Phi &= 0 & (7) \\ \Phi(\sqrt{\lambda} a) &= 0 & (8) \\ |\Phi(0)| &< \infty & (9) \end{aligned}$$

Equation (7) is a Bessel's equation of order zero, such that its general solution is expressed in terms of the Bessel's functions of order zero,

$$\Phi(z) = c_1 J_0(z) + c_2 Y_0(z)$$

Since we require Φ to be bounded at the origin, $c_2 = 0$. Then

$$\Phi(\sqrt{\lambda} a) = 0 \Rightarrow J_0(\sqrt{\lambda} a) = 0 \Rightarrow \lambda_n = \left(\frac{\mu_n^{(0)}}{a} \right)^2, \quad n = 1, 2, \ldots$$

where $\mu_n^{(0)}, n = 1, 2, \ldots$ denote the zeros of the regular Bessel's function $J_0(z)$. Then $\phi_n(r) = J_0(\sqrt{\lambda_n} r)$ and product solutions $u(r, t) = h(t)\phi(r)$ are of the form

$$u_n(r, t) = A_n \cos(c\sqrt{\lambda_n} t) J_0(\sqrt{\lambda_n} r) + B_n \sin(c\sqrt{\lambda_n} t) J_0(\sqrt{\lambda_n} r)$$

We seek for the solution to (1-3) as an infinite series

$$u(r,t) = \sum_{n=1}^{\infty} A_n \cos(c\sqrt{\lambda_n} t) J_0(\sqrt{\lambda_n} r) + B_n \sin(c\sqrt{\lambda_n} t) J_0(\sqrt{\lambda_n} r) \tag{10}$$

The coefficients a_n, b_n are obtained by imposing the boundary conditions (3):

$$u(r,\theta,0) = u_0(r) \Rightarrow \sum_{n=1}^{\infty} A_n J_0(\sqrt{\lambda_n} r) = u_0(r)$$

and using the orthogonality property

$$\int_0^a r J_0(\sqrt{\lambda_n} r) J_0(\sqrt{\lambda_m} r)\, dr = 0 \tag{11}$$

we get

$$A_n = \frac{\int_0^a r u_0(r) J_0(\sqrt{\lambda_n} r)\, dr}{\int_0^a r J_0^2(\sqrt{\lambda_n} r)\, dr}, \qquad n = 1, 2, \ldots \tag{12}$$

The second initial condition is used to find the coefficients B_n:

$$u_t(r,\theta,0) = v_0(r) \Rightarrow \sum_{n=1}^{\infty} B_n c\sqrt{\lambda_n} J_0(\sqrt{\lambda_n} r) = v_0(r) \Rightarrow B_n = \frac{1}{c\sqrt{\lambda_n}} \frac{\int_0^a r v_0(r) J_0(\sqrt{\lambda_n} r)\, dr}{\int_0^a r J_0^2(\sqrt{\lambda_n} r)\, dr}, \qquad n = 1, 2, \ldots \tag{13}$$

The solution to the problem (1-3) is thus expressed as the infinite series (10) with the coefficients given by (12-13).

2. Laplace's equation in a cylinder

We consider the Laplace equation

$$\nabla^2 u = 0$$

in a cylinder of height H and radius a. Introducing the cylindrical coordinates

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

the Laplace's equation is written

$$\frac{1}{r}[r u_r]_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz} = 0 \tag{14}$$

We assume that Dirichlet boundary conditions are prescribed on the top, bottom, and lateral surface of the cylinder:

$$u(r,\theta,H) = \beta(r,\theta) \quad (top) \tag{15}$$
$$u(r,\theta,0) = \alpha(r,\theta) \quad (bottom) \tag{16}$$
$$u(a,\theta,z) = \gamma(\theta,z) \quad (lateral\ \ boundary) \tag{17}$$

To find the solution, we split the problem (14-17) into three subproblems,

$$u = u_1 + u_2 + u_3$$

where each of u_1, u_2, u_3 satisfies only one nonhomogeneous boundary condition

$$u_1(r, \theta, H) = \beta(r, \theta)$$

$$u_2(r, \theta, 0) = \alpha(r, \theta)$$

$$u_3(a, \theta, z) = \gamma(\theta, z)$$

and takes zero values on the rest of the boundary. We used this approach before for the Laplace's equation in a rectangle. We search for product solutions of the form

$$u(t, \theta, z) = f(r)g(\theta)h(z) \tag{18}$$

such that, after replacing (18) into (14),

$$\frac{[rf']'}{rf} + \frac{1}{r^2}\frac{g''}{g} + \frac{h''}{h} = 0 \tag{19}$$

The z-variable may be first separated,

$$\frac{h''}{h} = -\left(\frac{[rf']'}{rf} + \frac{1}{r^2}\frac{g''}{g}\right) = \lambda$$

then,

$$\frac{r}{f}[rf']' + \lambda r^2 = -\frac{g''}{g} = \mu$$

Corresponding to $g(\theta)$ function we impose periodic boundary conditions,

$$g(-\pi) = g(\pi),\ g'(-\pi) = g'(\pi)$$

such that the eigenpairs (μ, g) are

$$\mu_m = m^2, g_m(\theta) = c_1 \cos(m\theta) = c_2 \sin(m\theta), \quad m = 0, 1, \ldots$$

For each of u_1, u_2, u_3 the differential equations

$$h'' = \lambda h \tag{20}$$

$$r[rf']' + (\lambda r^2 - m^2)f = 0 \tag{21}$$

must be solved with appropriate boundary conditions.

Subproblems 1 and 2

Notice that from the mathematical point of view the subproblems for u_1 and u_2 are quite similar, just flip the cylinder upside down. Then is enough to study the problem for u_1 which involves the Bessel's equation of order m:

$$r[rf']' + (\lambda r^2 - m^2)f = 0 \tag{22}$$

with boundary conditions

$$f(a) = 0$$

$$|f(0)| < \infty$$

From a previous analysis we know that $\lambda > 0$ and the eigenpairs (λ, f) are

$$\lambda_{mn} = \left(\frac{\mu_n^{(m)}}{a}\right)^2, \qquad f(r) = J_m(\sqrt{\lambda}_{mn} r), \quad n = 1, 2, \ldots \tag{23}$$

where $\mu_n^{(m)}, n = 1, 2, \ldots$ denote the zeros of the Bessel's function J_m.

The differential equation for h is

$$h'' = \lambda h \tag{24}$$

and since u_1 is zero on the bottom boundary $z = 0$, we impose $h(0) = 0$ such that, up to a multiplicative constant,

$$h(z) = \sinh(\sqrt{\lambda} z)$$

The solution $u1(r, \theta, z)$ is then expressed as a series

$$u_1(r, \theta, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sinh(\sqrt{\lambda}_{mn} z) J_m(\sqrt{\lambda}_{mn} r) \cos(m\theta) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sinh(\sqrt{\lambda}_{mn} z) J_m(\sqrt{\lambda}_{mn} r) \sin(m\theta) \tag{25}$$

where the coefficients A_{mn}, B_{mn} are determined from the boundary condition

$$u_1(r, \theta, H) = \beta(r, \theta)$$

Subproblem 3

For the u_3-subproblem the differential equation for h is

$$h'' = \lambda h \tag{26}$$

with boundary conditions

$$h(0) = 0, \qquad h(H) = 0$$

since u_3 takes zero values on the top and bottom boundaries of the cylinder. The eigenpairs (λ, h) are then

$$\lambda_n = -(n\pi/H)^2, \quad h_n(z) = \sin(n\pi z/H), \quad n = 1, 2, \ldots \tag{27}$$

such that solutions corresponding to both θ and z variables have an oscillatory behavior (sine and cosine functions). With λ_n above, the differential equation for the $r-$*dependent* solution becomes

$$r[rf']' + \left[-(n\pi/H)^2 r^2 - m^2\right] f = 0 \tag{28}$$

to which we must impose

$$|f(0)| < \infty$$

but there is no homogeneous condition at $r = a$. The change of variable

$$w = \frac{n\pi}{H} r, \quad F(w) = f(r)$$

may be used to transform (28) into a *modified Bessel's equation of order m*

$$w^2 F'' + wF'(-w^2 - m^2)F = 0 \tag{29}$$

which has a solution that is well defined at $w = 0$, the modified Bessel's function of order m of first kind, $I_m(w)$, and a solution that is singular at $w = 0$, the modified Bessel's function of order m of second kind, $K_m(w)$. Then

$$f(r) = c_1 K_m\left(\frac{n\pi}{H} r\right) + c_2 I_m\left(\frac{n\pi}{H} r\right)$$

and $|f(0)| < \infty$ implies $c_1 = 0$.

In conclusion, the solution u_3 is expressed as a double series

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} E_{mn} I_m\left(\frac{n\pi}{H} r\right) \sin\left(\frac{n\pi z}{H}\right) \cos(m\theta) + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} F_{mn} I_m\left(\frac{n\pi}{H} r\right) \sin\left(\frac{n\pi z}{H}\right) \sin(m\theta) \tag{30}$$

where the coefficients E_{mn}, F_{nm} are determined by imposing the boundary condition $u_3(a, \theta, z) = \gamma(\theta, z)$.