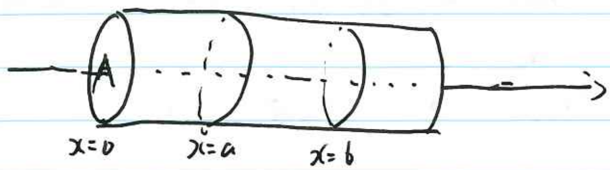
Heat equation.

Consider a 1D rod of length L



We want to derive an equation that governs the temperature distribution inside the rod.

First, we define a quantity $e(x,t)$ called thermal energy density which discribe the amount of thermal energy per unit volume.

Denote by A the cross-setional area of the rod and assume A is const.

Q1: What is the total thermal energy of the rod between $x=a$ and $x=b$?

$$\int_a^b e(x,t)\, A \, dx.$$

Q2 How can the total thermal energy be changed? We assume that the lateral surface to be insulated.

Conservation of heat energy:

rate of change of heat energy in time	=	heat energy flowing across boundaries per unit time	+	heat energy generated inside per unit time

Define: $\phi(x,t)$ = amount of thermal energy per unit time that flows to the right per unit surface area

We call $\phi(x,t)$ the heat flux.

Q3: What does $\phi(x,t) < 0$ mean?

Define $Q(x,t)$: heat energy per unit volume generated per unit time.

We call $Q(x,t)$ the heat sources.

Now, in equation, the conservation of heat energy reads.

$$\frac{d}{dt}\int_a^b e(x,t)\, A\, dx = \phi(a,t)\, A - \phi(b,t) + A\int_a^b Q(x,t)\,dx$$

Q4. Why on the LHS of the equation we ~~uses~~ use the ordinary differentiation $\frac{d}{dt}$ rather that $\frac{\partial}{\partial t}$?

Suppose $e(x,t)$ is differentiable.

$$\frac{d}{dt}\int_a^b e\, dx = \int_a^b \frac{\partial e}{\partial t}\, dx$$

If $\phi(x,t)$ is continously differentiable, by the fundamental theorem of calculus.

$$\phi(a,t)\, A - \phi(b,t)\, A = -A\int_a^b \frac{\partial \phi}{\partial x}\, dx$$

Hence, after dividing both sides and rearanging the terms

$$\int_a^b \left\{ \frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q \right] dx = 0 \quad \Rightarrow \text{integral conservation law.}$$

Since this holds for arbitrary a, b, $\left[\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q\right]$ must be zero.

Q5: Explain in detail why this is true? What happens if the integrand is not zero at somewhere?

Now we also have the differential conservation form

$$\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q = 0$$

Suppose now $Q = 0$.

If $\frac{\partial \phi}{\partial x} > 0 \Rightarrow \phi$ increases with $x \Rightarrow \phi(a,t) < \phi(b,t)$

$\Rightarrow \frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} < 0 \quad \Rightarrow e(x,t)$ will decrease in time.

Next, we want to look at the relation between thermal energy $e(x,t)$ and the temperature $u(x,t)$.

Define

$c(x)$. = heat capacity (the heat energy that must be supplied to a unit mass of a substance to raise its tempera one unit.

$\rho(x)$ = mass density (mass per unit volume).

~~Plug in the~~ The total mass of a thin slice is $\rho A \Delta x$, and hence the total thermal energy in a thin slice is $c(x)\, u(x,t)\, \rho A \Delta x$.

Therefore $\quad e(x,t)\, A\, \Delta x = c(x)\, u(x,t)\, \rho\, A\, \Delta x$

$$\Rightarrow \quad e(x,t) = c(x)\, \rho(x)\, u(x,t).$$

Plug this into the differential law of conservation.

$$c(x)\rho(x) \frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q \qquad (*)$$

Fourier's law of heat conduction

1. If the temperature is the same everywhere, there is no heat flow

2. If there is temperature variation, then heat flows from hot area to ~~the~~ cold area.
3. The greater the temperature difference, the greater heat flow.
4. Heat flow also depends on materials.

Fourier's law : $\phi = -K_0 \frac{\partial u}{\partial x}$
which asserts that the heat flux is proportional to the temperature difference, where $K_0 = K_0(x)$ is the thermal conductivity which measures the ability of the material to conduct heat.

Q6: Explain why there is a minus sign?

Suppose $\frac{\partial u}{\partial x} > 0 \Rightarrow$ u increases as x increases, hence $u(a,t) < u(b,t)$ and heat will flow from right to the left.

Now we are ready to derive the heat equation in 1D.
Put the Fourier's law in eqn $(*)$.

$$c\rho \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(-K_0 \frac{\partial u}{\partial x}\right) + Q$$

$$= \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q$$

In some special cases, c, ρ, K_0 are constants, we can further simplify the above equation.

$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q$$

If in addition, there are no sources, $Q = 0$

$$\Rightarrow \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad (**). \text{ where } k = \frac{K_0}{c\rho}$$

is called the thermal diffusivity.

Eqn. (**) is called the heat equation, which corresponds to no sources and constant thermal properties.

Initial / Boundary conditions.

Initial condition (IC) for heat equations

$$u(x,0) = f(x).$$

In order to predict the future temperature, only knowing initial condition is not enough. We need boundary conditions (BC) as well.

Q7: How many boundary condition do we need for heat equations?

There are 3 types of boundary conditions (BC).

1. Dirichlet BC : prescribed temperature.

For example at $x=0$, $u(0,t) = u_B(t)$, where $u_B(t)$ is a given temperature of surrounding medium.

2. Neumann BC: prescribed gradient / flux (Insulated boundary).

Eg. $$\underset{\uparrow}{-K_0(0)\frac{\partial u}{\partial x}\Big|_{x=0}} = \phi(t).$$ where $\phi(t)$ is given.

heat flux at $x=0$. If $\phi(t)=0 \Rightarrow$ no heat flux / perfectly insulated.

3. Mixed BC: Newton's law of cooling

Eg. At $x=0$.

$$\underbrace{-K_0(0)\frac{\partial u}{\partial x}\Big|_{x=0}}_{\text{heat flux}} = -H\,\underbrace{[u(0,t) - u_B(t)]}_{\substack{\text{difference of temperature}\\ \text{of the rod at } x=0 \text{ and}\\ \text{temperature } u_B \text{ of the surrounding}\\ \text{medium.}}}.$$

Here $H > 0$: convection coefficient / heat transfer coefficient.

at $x = L$ $\quad -K_0(L) \frac{\partial u}{\partial x}\Big|_{x=L} = H\,[u(L,t) - u_B(t)]$.

Q.8: Why the RHS of two boundary conditions should have different signs?

Remark: if $H \neq 0$, then this BC models non-perfectly insulated boundary, which mean the heat is lost or gained through the boundary ~~until~~ $u(x,t) \to u_B(t)$.