LEC 2

Last time: Derivation of the 1-D heat equation.

- conservation of energy $\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q = 0$
- Fourier's law of heat conduction $\phi = -K_0 \frac{\partial u}{\partial x}$.

General form: $c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q$

Special case: ~~if then~~ no sources, constant thermal property

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}} \quad \text{where } k = \frac{K_0}{c\rho}$$

Initial conditions: $u(x,0) = f(x)$.

Boundary conditions:

1. Prescribed temperature e.g. $u(0,t) = \underbrace{u_B(t)}_{\text{given}}$.
2. Prescribed heat flux
   e.g. at $x=0$
   $-K_0(0) \frac{\partial u}{\partial x}\Big|_{x=0} = \phi(t)$
3. Mixed BC.
   e.g. at $x=0$
   $-K_0(0) \frac{\partial u}{\partial x}\Big|_{x=0} = -H[u(0,t) - \underbrace{u_B(t)}_{\text{given}}]$

~~Stat~~. Steady-state temperature distribution

Let's consider the heat equation $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$, $0 \le x \le L$

Suppose we have the following initial and boundary conditions.

$$\begin{cases} u(x,0) = f(x) & \text{IC} \\ u(0,t) = T_1 & \\ u(L,t) = T_2 & \end{cases} \quad \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{BCs}$$

Def: A steady-state solution is a temperature distribution that does not depend on time, i.e. $u(x,t) = u(x)$.

By the definition, $\frac{\partial u}{\partial t} = 0$ hence the equation becomes

$$k \frac{\partial^2 u}{\partial x^2} = 0$$

Actually, this is equivalent to

$$\frac{d^2 u}{dx^2} = 0$$

since u is now a function of one variable.

The general solution of this ODE is

$$u(x) = C_1 x + C_2$$

where C_1, C_2 are some constants.
Since we are considering $u(x)$ not as a function in time, the initial condition can be ignored.
We have two BCs $\begin{cases} u(0) = T_1 \\ u(L) = T_2 \end{cases}$

Put these in $u(x) = C_1 x + C_2$, we get

$$\begin{cases} T_1 = C_2 \\ T_2 = C_1 L + C_2 \end{cases}$$

$$\text{Hence} \quad u(x) = T_1 + \frac{T_2 - T_1}{L} x$$

Now, if we consider the time-dependent problem, $u(x,t)$ will change in time. In the long run, the influence of the two ends will dominate the heat flow. Eventually, we expect the temperature approaches the equilibrium temperature distribution, i.e.

$$\lim_{t \to \infty} u(x,t) = u_E(x)$$

<u>Note</u>: 1) The IC has no effect on this ~~stat~~ steady-state solution.
2) We have used the fact that T_1, T_2 are constants.

Next, let's look at insulated boundaries.
We have the following time-dependent problem.

$$\begin{cases} \dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = f(x) \\ \dfrac{\partial u}{\partial x}(0,t) = 0 \\ \dfrac{\partial u}{\partial x}(L,t) = 0 \end{cases}$$

Again, we set $\frac{\partial u}{\partial t} = 0$, then out broblem becomes

$$\begin{cases} \dfrac{d^2 u}{dx^2} = 0 \\ \dfrac{du}{dx}(0) = 0 \\ \dfrac{du}{dx}(L) = 0 \end{cases}$$

Similar as previous calculation, $u(x) = C_1 x + C_2$

BCs $\Rightarrow$ C_1 must be zero $\Rightarrow u(x) = C_2$.

Q1: Is C_2 an arbitrary constants?

For the time-dependent problem, we expect

$$\lim_{t \to \infty} u(x,t) = C_2$$

Observation: the boundaries are perfectly insulated, which implies that the total thermal energy is ~~conse~~ a constant since there are no sources.

In fact,

$$\frac{d}{dt}\overbrace{\int_0^L c\rho u \, dx}^{\text{total thermal energy}} = -K_0 \frac{\partial u}{\partial x}(0,t) + K_0 \frac{\partial u}{\partial x}(L,t) = 0$$

Therefore the initial total energy must be equal to the equilibrium total energy

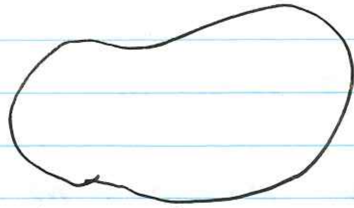
$$\int_0^L c\rho \, u(x,0) \, dx = \int_0^L c\rho \, C_2 \, dx$$

$$\Rightarrow \qquad C_2 = \frac{1}{L}\int_0^L f(x) \, dx = u(x).$$

Note: this tells us that the equilibrium solution is the average of the initial temperature distribution.

Heat Equations in higher dimensions

Consider a region R in 3D



R: region

∂R: boundary of R

$\hat{n}$: outward normal vector to the boundary.

The total thermal energy in R is

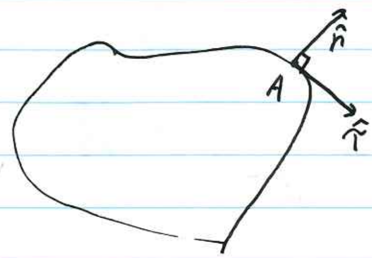
$$\iiint_R c(\vec{x})\rho(\vec{x})\, u(\vec{x},t)\, dV$$

Conservation of energy:

Rate of change of heat energy = heat energy flowing across boundaries per unit time + heat energy generated inside per unit time.

$$\text{LHS} = \frac{d}{dt} \iiint_R c\rho u \, dV$$

Let's look at the heat flux. Let $\vec{\phi}$ be the heat flux vector field which represents the heat energy flows per unit time in each unit area.



The ~~amount~~ magnitude of $\hat{\phi}$ is is the amount of heat energy flowing per unit time per unit surface area.

In fact, only the normal component of the heat flux contribute to the heat transfer. Let $\hat{\tau}$ be a tangent vector, then there will be no heat energy crossing A if the heat flux is in the direction of $\hat{\tau}$.

The outward normal component of $\vec{\phi}$ at A. is

$$\vec{\phi} \cdot \hat{n} = |\vec{\phi}| \cdot |\hat{n}| \cdot \cos\theta = |\vec{\phi}| \cos\theta$$

since $|\hat{n}| = 1$ and θ is the angle between the vector $\vec{\phi}$ and $\hat{n}$ at A.

The total amount of energy flowing across the boundary ∂R is then

$$\oiint_{\partial R} \vec{\phi} \cdot \hat{n} \, dS.$$

Hence the conservation ~~law~~ of energy is

$$\frac{d}{dt} \iiint_R c\rho u \, dV = - \oiint_{\partial R} \vec{\phi} \cdot \hat{n} \, dS + \iiint_R Q \, dV.$$

<u>Divergence theorem.</u>

Let $\vec{F}(x,y,z) = (f, g, h) = f\hat{i} + g\hat{j} + h\hat{k}$.

We can regard gradiant as an operator

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

Then $\nabla \cdot F = \operatorname{div}\vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle f, g, h \rangle$

$$= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

<u>Divergence theorem</u>

Suppose the vector field $\vec{F}$ is continuously differentiable, then

$$\iiint_R \nabla \cdot \vec{F}\, dV = \oiint_{\partial R} \vec{F} \cdot \hat{n}\, dS.$$

Note: in 1D, we have $-\int_a^b \frac{\partial \phi}{\partial x}\, dx = \phi(a) - \phi(b)$.

Using the divergence theorem,

$$\iiint_R c\rho \frac{\partial u}{\partial t}\, dV = -\iiint_R \nabla \cdot \vec{\phi}\, dV + \iiint_R Q\, dV$$

$$\Rightarrow \iiint_R \left(c\rho \frac{\partial u}{\partial t} + \nabla \cdot \vec{\phi} - Q\right) dV = 0.$$

Since R is an arbitrary region,

$$c\rho \frac{\partial u}{\partial t} + \nabla \cdot \vec{\phi} - Q = 0.$$

Fourier's law of heat conduction $\vec{\phi} = -K_0(\vec{x})\nabla u$

Heat equation.

$$c\rho \frac{\partial u}{\partial t} = \nabla \cdot (K_0 \nabla u) + Q$$

Special case ; no sources ~~Et~~, constant thermal properties.

$$\frac{\partial u}{\partial t} = \cancel{k\nabla^2 u}\; k\nabla\cdot(\nabla u) = k\Delta u .$$

Note: $\nabla\cdot(\nabla u) = \Delta u = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right\rangle$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

is called the Laplacian of u.

IC / BCs.

1. IC. $u(x,y,z,0) = f(x,y,z)$. initial temperature.
2. BCs.

1) Prescribed temperature.

$$u(x,y,z,t) = T(x,y,z,t) \quad \text{for } (x,y,z)\in\partial R.$$

Note: T is a function defined only on the boundary ∂R.

2) Prescribed heat flux.

$$-K_0 \nabla u \cdot \hat{n} = \phi(x,y,z,t) \quad \text{for } (x,y,z)\in\partial R.$$

Insulated boundary $\nabla u\cdot\hat{n} = 0$

Note: We write $\nabla u\cdot\hat{n} = \frac{\partial u}{\partial n}$ which is the directional derivative in the direction of $\hat{n}$.

3) Mixed BCs

$$-K_0 \nabla u\cdot\hat{n} = H(u - u_B) \quad , (x,y,z)\in\partial R$$

where u_B is the temperature of the surrounding medium. $H > 0$ is the convection coefficient.

Q. Compare this with the 1D case.

Steady-state solutions

Let $\frac{\partial u}{\partial t} = 0 \Rightarrow \nabla \cdot (K_0 \nabla u) + Q = 0$

If $K_0 \equiv$ const, then

$$\Delta u = -\frac{Q}{K_0}$$

this is the Poisson equation.

If in addition $Q = 0$, we have

$$\Delta u = 0$$

this is the Laplace equation.

Ch2. Method of Separation of variables.

2.1 Linearity.

<u>Def</u>: An operator L is <u>linear</u> if

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$$

where c_1, c_2 are arbitrary constants and u_1, u_2 are arbitrary functions.

<u>Ex</u> Let $L = \frac{\partial}{\partial t}$

$$L(c_1 u_1 + c_2 u_2) = \frac{\partial}{\partial t}(c_1 u_1 + c_2 u_2) = c_1 \frac{\partial u_1}{\partial t} + c_2 \frac{\partial u_2}{\partial t}$$

$$= c_1 L(u_1) + c_2 L(u_2).$$

Hence differentiation is a linear operator.

<u>Exercise:</u> Check that $L = \frac{\partial^2}{\partial x^2}$ is also a linear operator.

<u>Prop</u>. Any linear combination of linear operators is a linear operator.
Pf: HW.

Define the <u>heat operator</u> $L = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$. then L is linear since both $\frac{\partial}{\partial t}$, $k\frac{\partial^2}{\partial x^2}$ are linear.

<u>Def</u>: A linear equation for an unknown function u has the form

$$L(u) = f.$$

Hence the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q$ or

we can write $\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = Q$
is a linear partial differential equations.

<u>DEF</u> The linear PDE $Lu = f$ is <u>homogeneous</u> if $f = 0$.
Otherwise, it is nonhomogeneous. (i.e. $f \neq 0$).

<u>Principle of Linear superposition.</u>

If u_1 and u_2 satisfy the same linear homogeneous PDE, then their linear combination is also a solution to that PDE.

Pf: By assumption, $L(u_1) = 0 = L(u_2)$. where L is the given differential operator corresponding to the PDE. Then

$$L(c_1u_1 + c_2u_2) = c_1\cancel{L(u_1)} + c_2\cancel{L(u_2)} = 0.$$

Hence $(c_1u_1 + c_2u_2)$ is also a solution to $L(u) = 0$.

<u>Boundary conditions.</u>

Example of linear BCs.

$$\begin{cases} u(0,t) = f(t) \\ u_x(L,t) = g(t) \end{cases} \rightarrow \text{linear nonhomogeneous.}$$

$$\begin{cases} u_x(0,t) = 0 \\ u_x(L,t) = 0 \end{cases} \rightarrow \text{linear homogeneous.}$$