LEC 3. Heat equation with zero temperature at finite ends.

Consider the linear homogeneous 1D heat equation w/ homogeneous BCs

$$\begin{cases} \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} & 0 \leq x \leq L \\ u(0,t) = 0 = u(L,t) \\ u(x,0) = f(x) \end{cases}$$

Separation of variables

Look for solutions of the form

$$u(x,t) = \phi(x)\,G(t).$$

this is called the separation of variables, which allows us to reduce a PDE to ODE.

Plug the separation into original equation

$$\frac{\partial u}{\partial t} = \phi(x)\frac{dG}{dt} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{d^2\phi}{dx^2}G(t).$$

Hence

$$\phi(x)\frac{dG}{dt} = k\frac{d^2\phi}{dx^2}G(t)$$

$$\Rightarrow \underbrace{\frac{\frac{dG}{dt}}{kG(t)}}_{\text{function of } t \text{ only}} = \underbrace{\frac{\frac{d^2\phi}{dx^2}}{\phi(x)}}_{\text{function of } x \text{ only}} = -\lambda$$

$\Rightarrow$ both of them must be a constant.

here λ is the separation constant. The minus sign is introduced for later convenience.

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Let's temporarily ignore IC now.

Now we have 2 ODEs.

$$\frac{\phi''}{\phi} = -\lambda \Rightarrow \phi'' + \lambda\phi = 0.$$

$$\frac{G'}{kG} = -\lambda \Rightarrow G' + \lambda k G = 0.$$

The goal is to construct a <u>nontrivial</u> solution.

Note: $u(x,t) \equiv 0$ is always a solution to our equation, and we call this solution the <u>trivial solution</u>.

From the boundary conditions.

$$u(0,t) = 0 \Rightarrow \phi(0) \underbrace{G(t)}_{\not\equiv 0} = 0 \quad \Rightarrow \phi(0) = 0.$$

Here $G(t) \not\equiv 0$ means $G(t)$ cannot always be 0, since $G(t) \equiv 0$ implies $u(x,t) = 0 \cdot \phi(x) = 0$ which is trivial.

Similarly, $u(L,t) = 0 \Rightarrow \phi(L) \underset{\not\equiv 0}{G(t)} = 0 \quad \Rightarrow \phi(L) = 0.$

① The time-dependent problem.

$$G' + \lambda k G = 0$$

$$\frac{dG}{dt} + \lambda k G = 0 \Rightarrow \frac{dG}{G} = -\lambda k \, dt$$

$$\Rightarrow \ln|G(t)| = -\lambda k t + \tilde{C}$$

$$\Rightarrow G(t) = C e^{-\lambda k t}$$ where C is some constant.

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② Boundary value problem for $\phi(x)$.

$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(0) = 0 \\ \phi(L) = 0. \end{cases}$$

This is an <u>eigenvalue problem</u>, λ is called the <u>eigenvalue</u>. $\phi(x) \not\equiv 0$ is the <u>associated eigenfunction</u>

First consider $\lambda > 0$. The general solution is

$$\phi = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x.$$

(sometime we used $e^{i\sqrt{\lambda}x}$ and $e^{-i\sqrt{\lambda}x}$).

Apply the BCs.

$\phi(0) = 0 \Rightarrow \phi(0) = C_1 \cos 0 + C_2 \sin 0 = C_1 = 0$

$\phi(L) = 0 \Rightarrow \phi(L) = C_2 \sin(\sqrt{\lambda}L) = 0$

Either $C_2 = 0$ or $\sin(\sqrt{\lambda}L) = 0$. But $C_2 = 0$ will implie $\phi(x) = 0$ which is trivial. Hence $\sin(\sqrt{\lambda}L) = 0$.

$\Rightarrow \sqrt{\lambda}L = n\pi$ for $n > 0$.

Q: Why $n > 0$ here?

Therefore, the eigenvalue λ are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \cdots.$$

The corresponding eigenfunctions are

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \cdots.$$

Next. suppose $\lambda = 0$.

$$\frac{d^2\phi}{dx^2} = 0 \Rightarrow \phi(x) = C_1 + C_2 x$$

$\phi(0) = 0 \Rightarrow C_1 = 0$

$\phi(L) = 0 \Rightarrow C_2 \cdot L = 0 \Rightarrow C_2 = 0$

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Hence $\lambda = 0$ will give a trivial solution $\phi \equiv 0$.
Therefore $\lambda = 0$ is not an eigenvalue.

Finally, consider ~~λ~~ $\lambda < 0$.
The general solution is

$$\phi(x) = C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x}$$

or we can write this as

$$\phi(x) = C_3 \cosh \sqrt{\lambda} x + C_4 \sinh \sqrt{\lambda} x$$

$$\phi(0) = 0 \Rightarrow C_3 \cosh 0 = 0 \Rightarrow C_3 = 0.$$

$$\phi(L) = 0 \Rightarrow C_4 \underbrace{\sinh \sqrt{\lambda} L}_{\neq 0} = 0 \Rightarrow C_4 = 0$$

$\Rightarrow$ We also can only have trivial solution.

Hence ~~nontrivial so~~ ~~our~~ all eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, 3$$

$$\phi_n(x) = \sin \frac{n\pi x}{L}.$$

Hence we have a product solution to the PDE

$$u(x,t) = B \sin \frac{n\pi x}{L} \cdot e^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad n = 1, 2, \cdots$$

By principle of superposition, we can take the linear combination for different λ.

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}$$

In order to determine B_n, we need to use the initial condition.

At $t=0$ $\quad u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$.

Claim: Any functions (with certain restrictions that we will discuss later) can be written as an infinite linear combination of $\sin \frac{n\pi x}{L}$.
This expansion is called the Fourier sine series.

Q: What kind of properties must $f(x)$ satisfy?

Now our task is to compute B_n for each n.

Orthogonality of sines.

Proposition

$$\boxed{\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} = \begin{cases} 0 & \text{if } n \neq m \\ \frac{L}{2} & \text{if } n = m \end{cases}}$$

~~This can be easily~~ The proof of this formula is left as an exercise.

Now, consider $\int_0^L f(x) \sin \frac{m\pi x}{L}\, dx$.

$$= \int_0^L \left(\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\right) \sin\frac{m\pi x}{L}\, dx$$

$$= \sum_{n=1}^{\infty} B_n \int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = B_m \cdot \frac{L}{2}$$

since for all other n's, the integrals are 0.

Hence $\quad \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2} \cdot B_m$

$$\Rightarrow B_m = \frac{2}{L} \int_0^L f(x) \sin\frac{m\pi x}{L}\, dx.$$

~~Note~~ ~~Note~~:

Othogonality of functions.

Recall, vectors $\vec{A} = (x_1, x_2, \cdots, x_n)$ and $\vec{B} = (y_1, \cdots, y_n)$ are <u>orthogonal</u> if

$$\vec{A} \cdot \vec{B} = x_1 y_1 + \cdots x_n y_n = \sum_{i=1}^{n} x_i y_i = 0.$$

<u>Def</u>: Two function $f(x)$ and $g(x)$ defined on $[0,L]$ are orthogonal on $[0,L]$ if

$$\int_0^L f(x) g(x)\, dx = 0$$

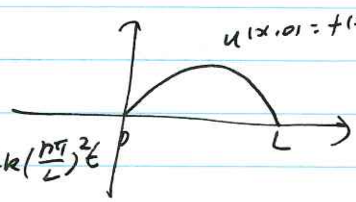
<u>Analogy</u>: - We can think of functions as infinite dimensional vectors

- integral $\int_0^L f(x) g(x)\, dx$ is like an infinite ~~vector~~ dimensional version of dot product $\vec{A} \cdot \vec{B}$

Ex. $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$, $t > 0$, $0 < x < L$

$u(0,t) = u(L,t) = 0$

$u(x,0) = x(L-x) = f(x)$.



We have $u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

where $B_n = \frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} f(x)\, dx$

$$= \frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} \cdot x(L-x)\, dx$$

(by computation)

$$= \frac{4L^2}{(n\pi)^3} \left(1 - (-1)^n\right)$$

Hence the solution for our Heat equation is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4L^2}{(n\pi)^3} \left[1-(-1)^n\right] \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Note: 1. $e^{-k\left(\frac{n\pi}{L}\right)^2 t}$ decays as $t \to \infty$

2. The first term ($n=1$) decays the slowest, followed by the term with $n=2$ etc.

3. We can get good approximation if we use only finite # of terms.

4. As t increases, we can use fewer terms to approximate our ~~an~~ solutions.

Heat conduction with insulated BCs.

Consider ~~$u_t = k u_{xx}$~~

$$\begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} & 0<x<L,\ t>0 \\ \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t) & \\ u(x,0) = f(x) & \end{cases}$$

Physically: perfectly insulated boundaries: no heat loss/gain at $x=0$ and $x=L$.

Separation of variables:

$u(x,t) = \phi(x)\cdot G(t)$.

$$\Rightarrow \quad G' + \lambda k G = 0 \quad \Rightarrow \quad G = Ce^{-\lambda k t}$$

$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi'(0) = 0 = \phi'(L) \end{cases}$$

Case I. $\lambda > 0$

general solution : $\phi(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$

$$\frac{d\phi}{dx} = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

$\frac{d\phi}{dx}(0) = 0 \Rightarrow C_2\sqrt{\lambda}\cdot 1 = 0 \Rightarrow C_2 = 0$. since $\lambda \neq 0$.

$\frac{d\phi}{dx}(L) = 0 \Rightarrow -C_1\sqrt{\lambda}\sin\sqrt{\lambda}L = 0$

since $C_1 \neq 0$, $\sin\sqrt{\lambda}L = 0$.

$\Rightarrow \sqrt{\lambda}L = n\pi$, $n = 1, 2, 3..$

$\Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, \cdots$

Here

$$\phi_n(x) = \cos\frac{n\pi x}{L}$$

Case II: $\lambda = 0$ $\phi(x) = C_1 + C_2 x$

$\frac{d\phi}{dx} = C_2 \Rightarrow C_2 = 0$ by BCs.

Therefore $\phi = C_1$ is an eigenfuntion for $C_1 \neq 0$.
Hence $\lambda = 0$ is an eigenvalue.

Case III : $\lambda < 0$ no eigenvalues. Left as an exercise.

~~Alle~~ All eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2, \cdots$$

(note we put $\lambda = 0$ case inside)

with ~~eigen~~ eigenfunctions.

$$\phi_n(x) = \cos\frac{n\pi x}{L} \quad n = 0, 1, 2, \cdots$$

Q : Why we can have $n = 0$ here?

Now ~~the~~ by superposition.

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t}$$

$$= \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t}$$

I.C. $u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}$ $\Rightarrow$ Fourier cosine series.

To find A_n, we need.

Orthogonality of cosines

Proposition $\int_0^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \neq 0 \\ L & n = m = 0. \end{cases}$

Then $\int_0^L f(x) \cos\frac{m\pi x}{L}\, dx = \sum_{n=0}^{\infty} A_n \int_0^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx$

$$= \begin{cases} A_m \cdot \frac{L}{2} & \text{if } n = m \neq 0 \\ A_m L & \text{if } n = m = 0. \end{cases}$$

Therefore $A_0 = \frac{1}{L} \int_0^L f(x)\, dx$

$$A_m = \frac{2}{L} \int_0^L f(x) \cos\frac{m\pi x}{L}\, dx, \quad m > 0$$

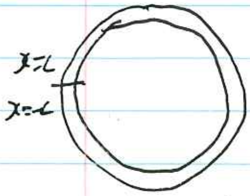
Now $u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t}$

$$= A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t}$$

Q: What is the stead-state of this solution, i.e.: $\lim_{t\to\infty} u(x,t)$?

In fact $\lim u = A_0 = \frac{1}{L}\int_0^L f(x)\,dx$ which is the average of the initial temperature distribution. This confirms our claim ~~to~~ before.

<u>Heat conduction in a thin ring (Periodic BCs).</u>



a thin circular ring

~~Assume~~ Consider heat flows in a circle ring with constant thermal properties and no sources. in x direction.
The heat distribution $u(x,t)$ can be think of a periodic function on the real line

Mathematically, we have periodic BCs.

$$\begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} & -L<x<L \quad t>0. \\ u(-L,t) = u(L,t) & \\ \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t) & \Big\} \text{ periodic BCs.} \\ u(x,0) = f(x). \end{cases}$$

(Here we assume both $u(x,t)$ and $\frac{\partial u}{\partial x}(x,t)$ are continuous)

Again, using the separation of variables.

$$u(x,t) = \phi(x)G(t).$$

$$\Rightarrow G(t) = Ce^{-\lambda kt}$$

Then
$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(-L) = \phi(L) \\ \phi'(-L) = \phi'(L). \end{cases}$$

<u>Case I:</u> $\lambda > 0$ gen. sol. $\phi(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$

$$\phi(-L) = \phi(L) \Rightarrow C_1 \underbrace{\cos(\sqrt{\lambda}(-L))}_{\cos\sqrt{\lambda}L} + C_2 \underbrace{\sin(\sqrt{\lambda}(-L))}_{-\sin\sqrt{\lambda}L} = C_1\cos\sqrt{\lambda}L + C_2\sin\sqrt{\lambda}L$$

Note: cos is an even function, sine is odd.

$$\Rightarrow 2C_2 \sin\sqrt{\lambda}L = 0$$

$\phi'(-L) = \phi'(L) \Rightarrow$

$$-C_1\sqrt{\lambda}\underbrace{\sin(\sqrt{\lambda}(-L))}_{-\sin(\sqrt{\lambda}L)} + C_2\sqrt{\lambda}\underbrace{\cos(\sqrt{\lambda}(-L))}_{\cos(\sqrt{\lambda}L)} = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) + C_2\sqrt{\lambda}\cdot\cos\sqrt{\lambda}L$$

$$\Rightarrow 2C_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$$

Since C_1, C_2 cannot both be 0. $\Rightarrow \sin\sqrt{\lambda}L = 0$

$$\Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \ldots$$

$$\phi_n(x) = c_n \cos\frac{n\pi x}{L} + d_n \sin\frac{n\pi x}{L}.$$

<u>case II:</u> $\lambda = 0$.

$\phi'' = 0 \Rightarrow \phi(x) = C_1 + C_2 x$.

$\phi'(x) = C_2 = 0$ since $\phi(-L) = \phi(L)$

$C_1 - C_2\cdot L = C_1 + C_2\cdot L \Rightarrow 2C_2\cdot L = 0.$

$\Rightarrow C_2 = 0.$

$\phi'(-L) = \phi'(L) = 0$. V. automatically satisfied.

C_1 can be arbitrary. Hence.

$\phi(x) = C_1 \neq 0$ is an eigenfunction. In particular, we can pick ~~$\phi(x)=$~~ $\phi(x) = 1$.

Case III: $\lambda < 0$ <u>left as an exercise.</u>

Remark: This is only one eigenfuncition associated w/ $\lambda = 0$. But for $\lambda_n > 0$, each λ_n has two eigenfunctions $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$.

By superposition, the general form of the solution is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t}$$

Initial conditions $u(x,0) = f(x)$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$ ↗ Fourier series

To solve for a_n, b_n, we need orthogonality for both sine and cosine.

1. $$\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 2L & n = m = 0 \end{cases}$$

2. $$\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \begin{cases} 0 & n \neq m \\ L & n = m \end{cases}$$

3. $$\int_{-L}^{L} \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = 0.$$

Pf: Exercise.

As before, by calculating $\int_{-L}^{L} f(x) \begin{Bmatrix} \cos \frac{m\pi x}{L} \\ \sin \frac{m\pi x}{L} \end{Bmatrix} dx$

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We get $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \quad m>0$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \quad m>0$$

Summary: BVP for $\phi'' + \lambda\phi = 0$

Dirichlet	Neumann	Periodic
$\phi(0) = \phi(L) = 0$	$\phi'(0) = \phi'(L) = 0$	$\begin{cases} \phi(0) = \phi(L) \\ \phi'(-L) = \phi'(L) \end{cases}$