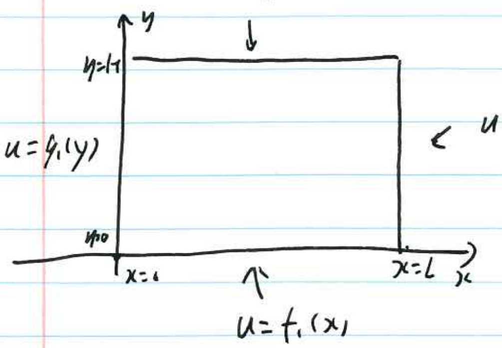
Laplace equation in 2D

1. Laplace equation in 2D

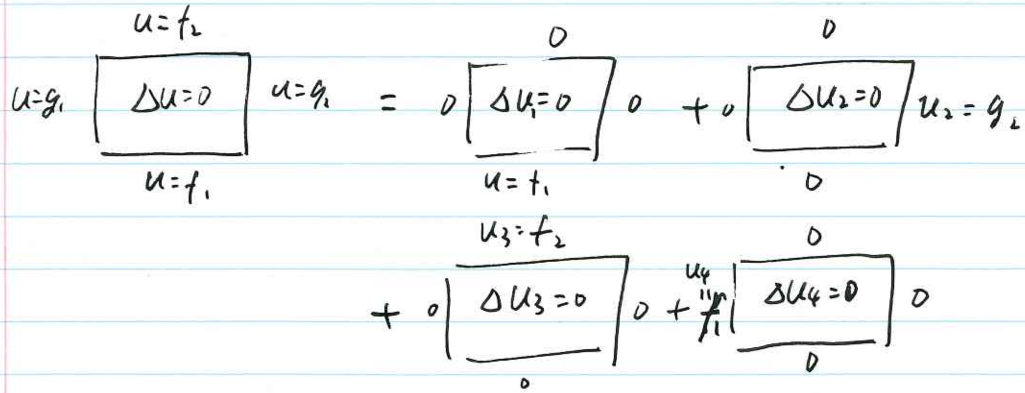


$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

BCs:
$$u(0,y) = g_1(y)$$
$$u(L,y) = g_2(y)$$
$$u(x,0) = f_1(x)$$
$$u(x,H) = f_2(x). \qquad (*)$$

Note: $\Delta u = 0$ is linear and homogeneous. but its BCs are not homogeneous. Hence we cannot apply separation of variables directly.

Consider the decomposition.



then ~~and~~ $u = u_1 + u_2 + u_3 + u_4$ satisfies PDE and BCs $(*)$ above.

The method for solving these 4 problem are similar. Let's look at u_4 problem only.

Our PDE and BCs are

$$\begin{cases} \Delta u_4 = \dfrac{\partial^2 u_4}{\partial x^2} + \dfrac{\partial^2 u_4}{\partial y^2} = 0 & 0 \le x \le L, \quad 0 \le y \le H \\ u_4(x,0) = 0 = u_4(x,H) \\ u_4(L,y) = 0 \\ u_4(0,y) = g_1(y). \end{cases}$$

Note: We have 3 homogeneous BCs and 1 non-hom BC.

Separation of variables.

$$u_4(x,y) = h(x)\,\phi(y).$$

For 3 homogeneous BCs, we have
$h(L)=0 \quad \phi(0)=0 \quad \phi(H)=0.$

Plug the separation into PDE

$$h''\phi + \phi'' h = 0 \qquad \text{divide both sides by } \phi h$$

$$\Rightarrow \quad \frac{h''}{h} = -\frac{\phi''}{\phi} = \lambda$$

Q: why we set the constant as λ rather than $-\lambda$ as in heat equation?

Y- problem

$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(0) = 0 = \phi(H) \end{cases}$$

Dirichlet BCs.
$\Rightarrow$ eigenvalues: $\lambda_n = \left(\frac{n\pi}{H}\right)^2$, $n=1,2,\cdots$
eigenfunctions $\phi_n = \sin\frac{n\pi y}{H}$

x - problem

For ~~h(y)~~ $h(x)$, we only have 1 BC.

$$\begin{cases} \dfrac{d^2h}{dx^2} - \lambda h = 0 \\ h(L) = 0 \end{cases}$$

note that ~~λ~~ from the eigenvalue problem. $\lambda_n = \left(\frac{n\pi}{H}\right)^2 > 0$

Hence solutions are combinations of $e^{\pm\sqrt{\lambda_n}x}$ or $\cosh\sqrt{\lambda_n}x$ and $\sinh\sqrt{\lambda_n}x$.

Since $h(L) = 0$, ~~as~~ $\cosh(x) \neq 0$ for all x and $\sinh x = 0$ if only if $x = 0$. Therefore, we consider

$$h(x) = C \sinh\left(\frac{n\pi}{H}(x-L)\right)$$

Product solution: $u_4(x,y) = A_n \sin\frac{n\pi y}{H} \sinh\left((x-L)\frac{n\pi}{H}\right)$

Remark: The solution oscillates in y but not in x. This is a typical property of the solutions of Laplace equations.

General form by superposition.

$$u_4(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{H} \sin\left(\frac{n\pi}{H}(x-L)\right)$$

Next, we need solve for A_n. Note that there is one non homogeneous BC $u_4(0,y) = g_1(y)$. We will use this as well as the orthogonality of sines to find A_n.

$$u_4(0,y) = g_1(y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{H} \underbrace{\sinh\left(\frac{n\pi}{H}(-L)\right)}_{\text{constants}}$$

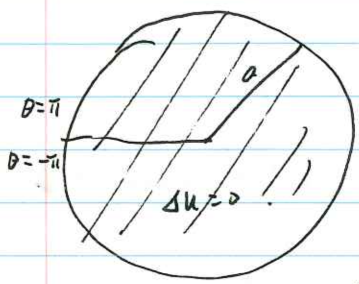
$$\Rightarrow \quad g_1(y) = \sum_{n=1}^{\infty} \underbrace{\left(A_n \sinh\left(-\frac{n\pi L}{H}\right)\right)}_{\text{Fourier coefficients.}} \sin\frac{n\pi y}{H}$$

Now. $\int_0^H g_1(y) \sin \frac{m\pi y}{H} dy = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi L}{H} \int_0^H \sin \frac{n\pi y}{H} \cdot \sin \frac{m\pi y}{H}$

$$= - A_m \sin \frac{m\pi L}{H} \cdot \frac{H}{2}$$

$$\Rightarrow A_m = - \frac{2}{H} \frac{1}{\sinh \frac{m\pi L}{H}} \int_0^H g_1(y) \sin \frac{m\pi y}{H} dy$$

Laplace equation on a circular disk



$-\pi \le \theta \le \pi \qquad 0 \le r \le a.$

$\Delta u = 0$ inside a circular disk gives a ~~stat~~ steady state solution for heat distribution.

- Assume constant thermal property and no sources.
- We have prescribed boundary temperature.

By the symmetry of the disk, we will use polar coordinates for simplicity.
In polar coordinates, ~~the laplacian of u is~~ the Laplace equation is

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.$$

Exercise: derive the above formula from the rectangular ~~Laplace~~ Laplace equation.

Our BC is $u(a, \theta) = f(\theta)$.
Also we assume $|u(0, \theta)| < \infty$: bounded at the origin.
Note that as in heat equation in a thin ring, we have the periodic BCs

$$\begin{cases} u(r,-\pi) = u(r,\pi) \\ \cancel{\text{th}} \ \frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi) \end{cases}$$

Separation of variables. $u(r,\theta) = \phi(\theta) G(r)$.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi(\theta) + \frac{1}{r^2}\frac{d^2\phi}{d\theta^2}G(r)$$

$$\underbrace{\frac{r\frac{d}{dr}\left(r\frac{dG}{dr}\right)}{G}}_{\text{function only in } r} = \underbrace{-\frac{\frac{d^2\phi}{d\theta^2}}{\phi}}_{\text{function only in } \theta} = \lambda$$

eigenvalue problem

$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(-\pi) = \phi(\pi) \\ \phi'(-\pi) = \phi(\pi) \end{cases}$$

As in heat equation on a ring, we set $L=\pi$. then

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 = \left(\frac{n\pi}{\pi}\right)^2 = n^2$$

$n = 0, 1, 2, \cdots$

eigenfunctions : $\begin{cases} \cos\frac{n\pi\theta}{L} = \cos\frac{n\pi\theta}{\pi} = \cos n\theta & n=0,1,\cdots \\ \sin\frac{n\pi\theta}{L} = \sin n\theta & n=1,2,\cdots \end{cases}$

note that $\lambda_0 = 0$ corresponds to the constant eigenfunction $\phi_0 = \cos 0\theta = 1$.

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r - problem

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) - \lambda G(r) = 0$$

$$r\left(r\frac{d^2G}{dr^2} + \frac{dG}{dr}\right) - \lambda G(r) = 0$$

$$r^2 G'' + rG' - \lambda G = 0$$ Equidimensional equation (Euler)

Also $|G(0)| < \infty$ from BCs.

Look for solutions of the form $G(r) = r^p$

$$G' = pr^{p-1} \quad \cancel{G'' = p(p-1)r} \quad G'' = p(p-1)r^{p-2}$$

$$\therefore \quad p(p-1)r^p + pr^p - \lambda r^p = (p^2 - p + p - \lambda)r^p$$

$$= (p^2 - \lambda)r^p = 0$$

Since $\cancel{\lambda \geq 0}$ $\Rightarrow$ $p = \pm n$.

$\lambda = n^2$ for $n \geq 0$

If $n \neq 0$ solutions are $G(r) = C_1 r^n + C_2 r^{-n}$

since $|G(0)| < \infty$, $C_2 = 0$, $G(r) = C_1 r^n$

If $n = 0$. We get only 1 independent solution $G = r^0$.

Since $\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \cancel{r^2G'' + rG'} = 0$.

$$\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0 \Rightarrow r\frac{dG}{dr} = \tilde{C}_2$$

$$\therefore \frac{dG}{dr} = \frac{\tilde{C}_2}{r} \Rightarrow G(r) = \tilde{C}_2 \ln r + \tilde{C}_1$$

Hence $\ln r$ is also a solution.

But since $\ln r$ is not bounded at the origin, $\tilde{C}_2 = 0$.

Hence.
$$G(r) = \begin{cases} C_1 r^n & n > 0 \\ \tilde{C}_1 & n = 0. \end{cases}$$

Therefore, by using products and superposition, the general form of $u(r,\theta)$ is

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta$$

Note: solution is oscillatory in θ but not in r.

Next, we need find A_n and B_n. We have one more BC at $r = a$.

$$u(a,\theta) = f(\theta) = \sum_{n=0}^{\infty} \underbrace{A_n a^n}_{} \cos n\theta + \sum_{n=1}^{\infty} \underbrace{B_n a^n}_{} \sin n\theta$$

Fourier coefficients.

Using the orthogonality of sines and cosines

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta \qquad n = 0.$$

$$\left.\begin{aligned} A_n a^n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta \\ B_n a^n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta \end{aligned}\right\} \quad n \geq 1$$

Ch3. Fourier series

Recall: Solution to the heat equation on the ring

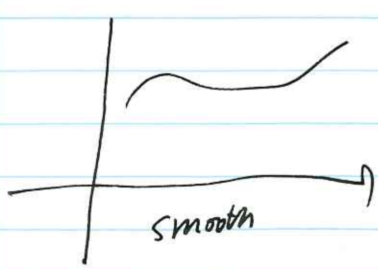
$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t}$$

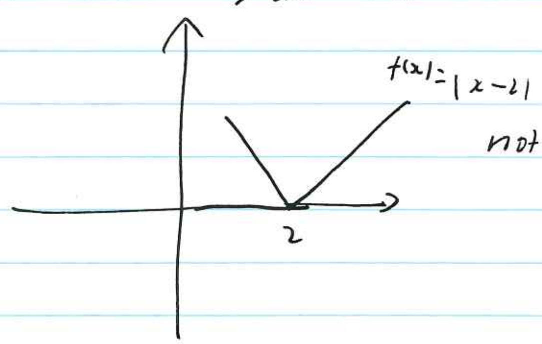
$$IC: u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

This is a Fourier series.

Does this infinite converge? If it does converges, will it converges to $f(x)$?

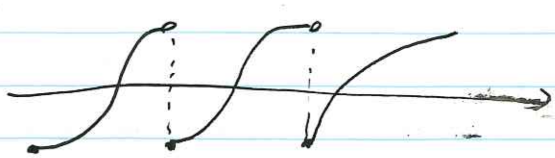
Def: A function $f(x)$ is piecewise smooth in $a < x < b$ if $[a,b]$ can be broken up into pieces such that in each piece both $f(x)$ and df/dx are continuous.





not smooth.

Ex.

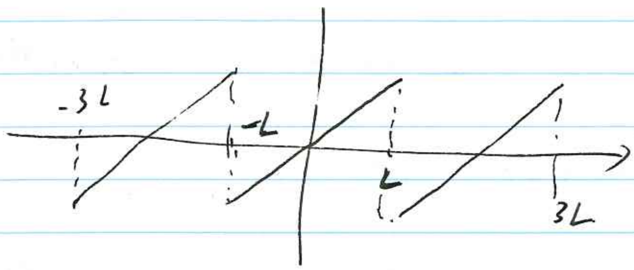


piecewise smooth.

Def: A function $f(x)$ has a jump discontinuity at $x = x_0$ if both the left limit $f(x_0^-) = \lim_{x \to x_0^-} f(x)$ and the right limit $f(x_0^+) = \lim_{x \to x_0^+} f(x)$ exists, but $f(x_0^+) \neq f(x_0^-)$.

Def: Let $f(x)$ be defined on $-L \le x \le L$. ~~The~~ We can define the periodic extension of $f(x)$ on the whole real line by translating the graph of $f(x)$ into the intervals $-L + 2nL \le x \le L + 2nL$, $n \in \mathbb{Z}$.

Ex. $f(x) = x$, $-L \le x \le L$.



Def: The <u>Fourier series</u> of a function $f(x)$ over an interval $-L \le x \le L$ is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

and the <u>Fourier coefficients</u> are

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \qquad n \ge 1.$$

Note: We use symbol "$\sim$" to say that $f(x)$ has a Fourier series, but this series may not converge, or if it converges, it may not converges to $f(x)$.

Fourier theorem

Let $f(x)$ be a piecewise smooth function on $[-L, L]$ then its Fourier series converges to

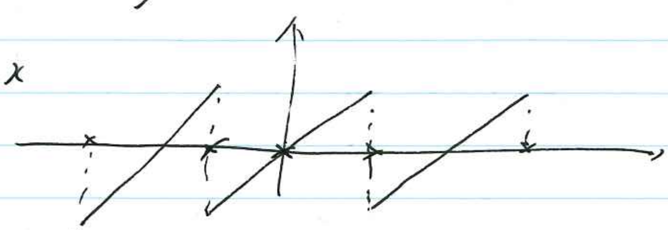
1). periodic extension of $f(x)$, where the periodic extension is continuous.

2) average value of the two limit

$$\frac{1}{2}\left[f(x_0^+) + f(x_0^-) \right]$$

where the periodic extension has a finite jump discontinuity.

Ex. $f(x) = x$



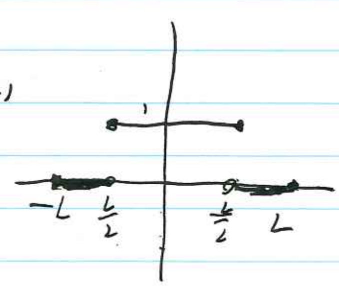
Sketching Fourier series

1. Sketch $f(x)$ on $(-L, L)$
2. Sketch the periodic extenison of $f(x)$.
3. Mark an "x" at the average of the two values at any ~~two~~ jump discontinuity.

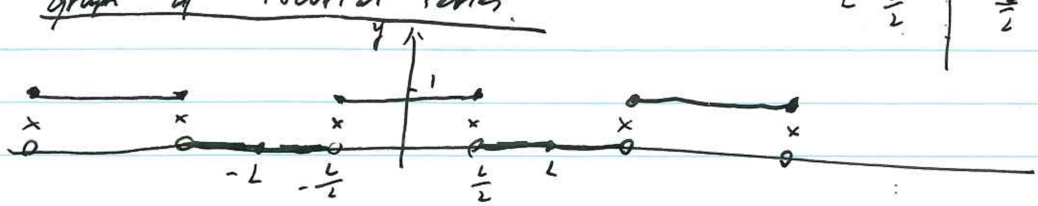
Ex. Sketch Fourier series for

$$f(x) = \begin{cases} 1 & |x| < \frac{L}{2} \\ 0 & |x| > \frac{L}{2} \end{cases}$$

$f(x)$



graph of Fourier series



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$$\therefore \; a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \begin{cases} 1, & x \in \left(-\frac{L}{2}, \frac{L}{2}\right) \\ 0, & x \in \left[-L, -\frac{L}{2}\right) \cup \left(\frac{L}{2}, L\right) \\ \frac{1}{2}, & x = \pm\frac{L}{2} \end{cases}$$

Let's compute the Fourier coefficients

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{2L}\int_{-L}^{-L/2} 0 \cdot dx + \frac{1}{2L}\int_{-\frac{L}{2}}^{\frac{L}{2}} 1\,dx + \frac{1}{2L}\int_{\frac{L}{2}}^{L} 0 \cdot dx = \frac{1}{2L}\cdot L = \frac{1}{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L} = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Bigg|_{x=-\frac{L}{2}}^{x=\frac{L}{2}} = \frac{1}{n\pi}\left(\sin\frac{n\pi}{L}\cdot\frac{L}{2} - \sin\frac{n\pi}{L}\cdot\left(-\frac{L}{2}\right)\right)$$

$$= \frac{2}{n\pi}\cdot\sin\frac{n\pi}{2}$$

n odd $\Rightarrow n = 2k+1 \Rightarrow \sin\frac{n\pi}{2} = (-1)^k$

n even $\Rightarrow n = 2k \Rightarrow \sin\frac{n\pi}{2} = \sin k\pi = 0.$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \sin\frac{n\pi x}{L}\,dx$$

$$= -\frac{1}{L}\cdot\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Bigg|_{x=-\frac{L}{2}}^{x=\frac{L}{2}} = 0$$

Hence $$f(x) \sim \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{L}$$

or $$f(x) \sim \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi}\cdot(-1)^k \cos\frac{(2k+1)\pi x}{L}.$$