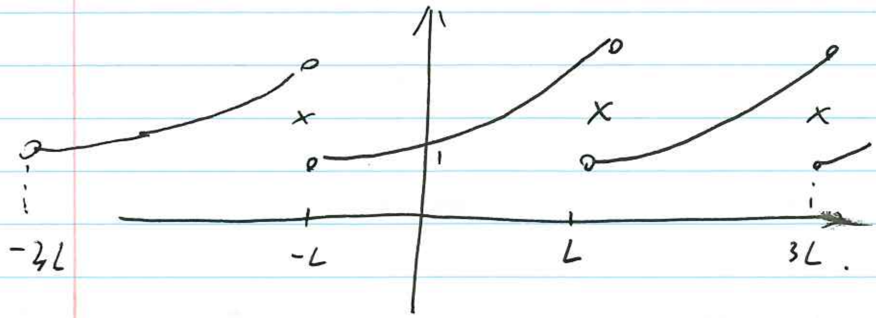
LEC 5 (First part is at the end of last notes)

Ex. $f(x) = e^x$. Sketch Fourier series.



Q: Compute the Fourier coefficients.

Fourier sine and cosine series

DEF. An odd function has a property $f(-x) = -f(x)$.

Ex. $f(x) = \sin x$, $f(x) = x$, $f(x) = x^3$ etc.

Fourier coefficients.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{0} f(x)\,dx + \frac{1}{2L}\int_0^L f(x)\,dx$$

$$= \frac{1}{2L}\int_{0}^{L} -f(x)\,dx + \frac{1}{2L}\int_0^L f(x)\,dx = 0.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{odd}}\,\underbrace{\cos\frac{n\pi x}{L}}_{\text{even}}}_{\text{odd}}\,dx = 0 \qquad n = 1, 2, \cdots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{odd}}\,\underbrace{\sin\frac{n\pi x}{L}}_{\cancel{\text{even}}\ \text{odd}}}_{\text{even}}\,dx = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$\therefore \quad f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$: Fourier sine series.

<u>Recall:</u> solutions to $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ with Dirichlet BCs on $0<x<L$ is

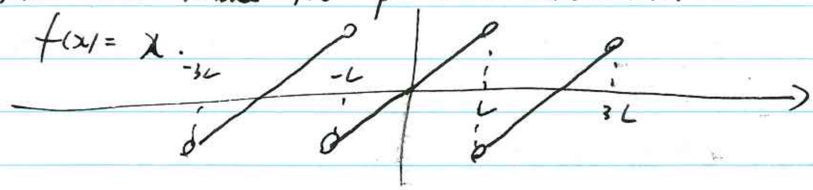
$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-k(\frac{n\pi}{L})^2 t}$$

w/ IC : $u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x)$

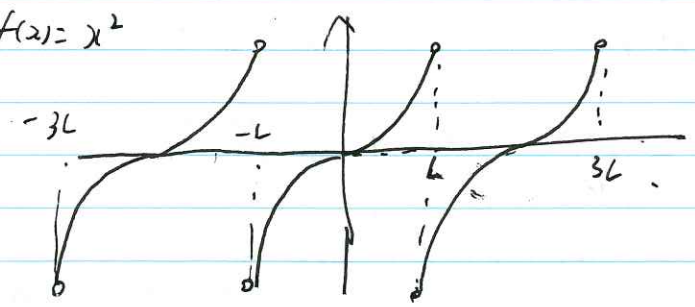
<u>Note:</u> $f(x)$ is any function on $[0,L]$, and its Fourier series is an odd function on $[-L,L]$.

<u>Def:</u> Let $f(x)$ be a function on $0<x<L$. The odd extension of $f(x)$ is obtained by defining $-f(x) = f(-x)$ for $x \in (-L, 0)$ and let $f(0)=0$. And then make the periodic extension.

Ex. $f(x) = x$.



Ex. $f(x) = x^2$



$\Rightarrow$ The odd extension of $f(x)$ has Fourier sine series.

odd extension of $f(x) \sim \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$.

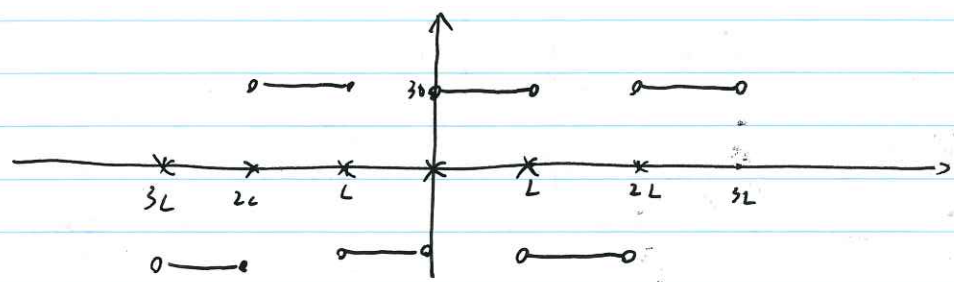
<u>Note:</u> The convergence theorem (Fourier theorem)

applies to the Fourier sine series as well.

Sketching Fourier sine series

1. Sketch function $f(x)$ only on $0 \le x \le L$
2. Sketch odd extension of $f(x)$
3. Mark "x" ~~at the~~ the average where the odd extension of $f(x)$ has a finite jump discontinuity.

Ex. Sketch the Fourier sine series for $f(x) = 30$.



Exercise: Compute B_n for $f(x) = 30$.

Next, let's look at even functions

DEF: ~~Even functions for~~
~~Let~~ An even function is a function $f(x)$ with function $f(x) = f(-x)$.

Note: In fact, $f(x)$ is symmetric w.r.t. y-axis.

Let's look at the Fourier coefficients.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} \underbrace{f(x)}_{\text{even}}\,dx = \frac{1}{2L}\int_0^L f(x)\,dx + \frac{1}{2}\int_{-L}^{0} f(x)\,dx$$

$$= \frac{1}{L}\int_0^L f(x)\,dx.$$

$$a_n = \frac{1}{\cancel{L}}\int_{-L}^{L} \underbrace{f(x)}_{\text{even}} \underbrace{\cos\frac{n\pi x}{L}}_{\text{even}} = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{even}}\ \underbrace{\sin\frac{n\pi x}{L}}_{\text{odd}}}_{\text{odd}}\,dx = 0.$$

Hence $f(x) \sim a_0 + \sum_{n\geq 1}^{\infty} a_n \cos\frac{n\pi x}{L} = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L}$

$\Rightarrow$ Fourier coefficients.

Recall: Solution of $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ on $0 < x < L$ w/ insulated boundaries has the form

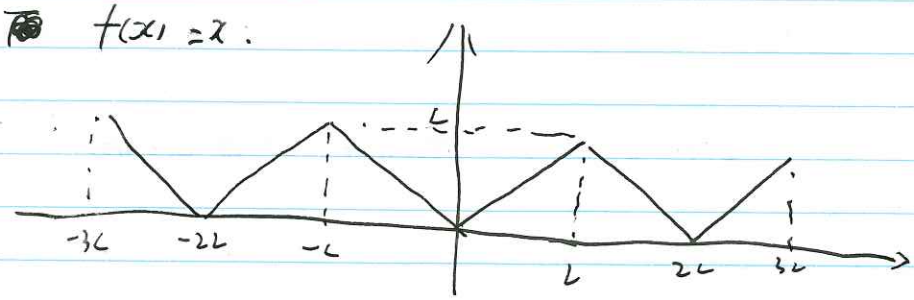
$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

w/ IC: $u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}$.

Note: $f(x)$ need not be even but its Fourier series is even.

We can define the even extension by similar construction as we did in odd extension.

Ex. $f(x) = x$.



Note: Even periodic extension is automatically continuous at $x = 0, \pm L, \pm nL, \cdots$

Then

the even extension of $f(x)$ $\sim \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}$

where $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$

$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.$

Sketch Fourier cosine series

1. Sketch $f(x)$ only on $0 \le x \le L$
2. Make the even periodic extension.
3. Mark an "x" ~~at each~~ the average of two limits at each jump discontinuity.

How does the Fourier series, Fourier sine series, Fourier cosine series relate?

Claim: Any function can be written as a sum of an even and an odd functions.

Pf:

Let $f(x) = \underbrace{\frac{1}{2}[f(x) + f(-x)]}_{f_e(x)} + \underbrace{\frac{1}{2}[f(x) - f(-x)]}_{f_o(x)}$.

even part of $f(x)$ odd part of $f(x)$.

Exercise: Verify $f_e(x)$ is even and $f_o(x)$ is odd.

Ex. $f(x) = \frac{1}{x+1}$

$$f_e(x) = \frac{1}{2}[f(x) + f(-x)] = \frac{1}{2}\left[\frac{1}{x+1} + \frac{1}{-x+1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{x+1} - \frac{1}{x-1}\right] = \frac{1}{2}\,\frac{x-1-x-1}{(x+1)(x-1)} = -\frac{1}{x^2-1}$$

$$f_o(x) = \frac{1}{2}[f(x) - f(-x)] = \frac{1}{2}\left[\frac{1}{x+1} - \frac{1}{-x+1}\right]$$

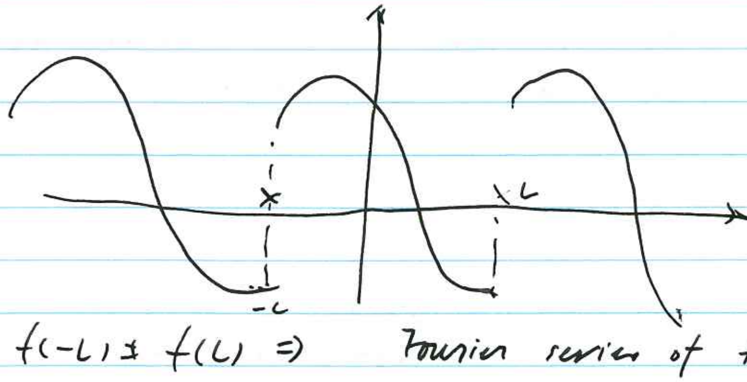
$$= \frac{1}{2}\left[\frac{1}{x+1} + \frac{1}{x-1}\right] = \frac{1}{2}\,\frac{2x}{(x+1)(x-1)} = \frac{x}{x^2-1}$$

THM Fourier series of $f(x)$ equals Fourier cosine series of the even part $f_e(x)$ of $f(x)$ plus Fourier sine series of the odd part $f_o(x)$ of $f(x)$.

Convergence of Fourier series

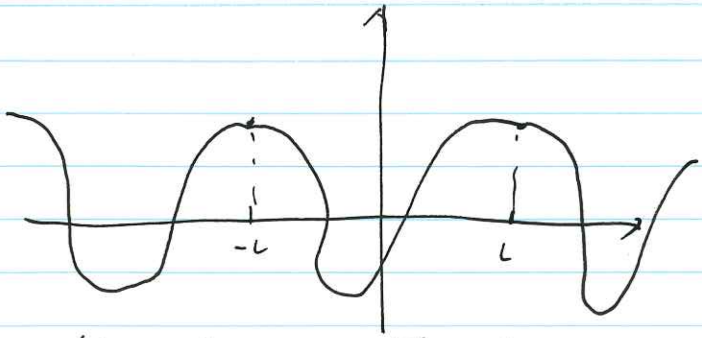
Thm 1. The Fourier series of $f(x)$ is continuous and converges to $f(x)$ in $-L \le x \le L$ if and only if $f(x)$ is continuous and $f(-L) = f(L)$.

Ex.



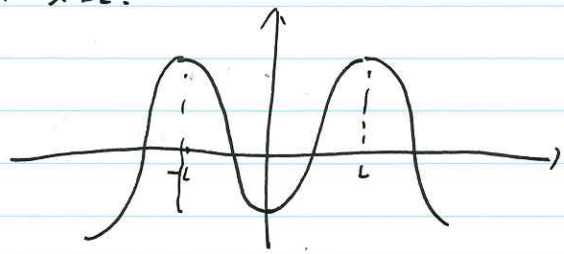
$f(-L) \neq f(L) \Rightarrow$ Fourier series of $f(x)$ doesn't converges to $f(x)$ at $x = (2n+1)L$.

Ex.



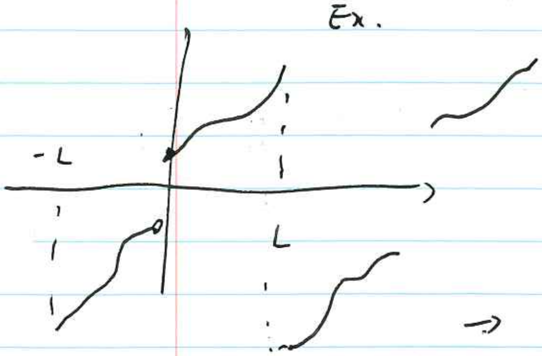
$f(-L) = f(L) \Rightarrow$ The Fourier series converges to $f(x)$ on $-L \le x \le L$

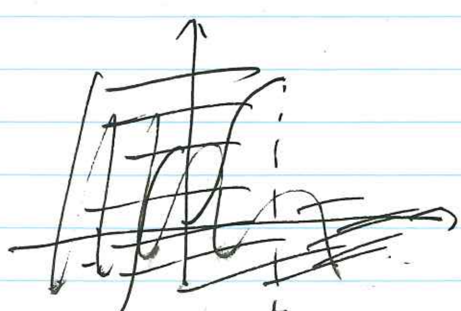
THM 2. The Fourier cosine series of $f(x)$ is continuous and converges to $f(x)$ for $0 \le x \le L$ if and only if $f(x)$ is continuous. (Note that the even extension guarantees ~~the~~ the continuity at $x=0$ and $x=L$.



THM 3. The Fourier ~~cosine~~ sine series of $f(x)$ is continuous and converges to $f(x)$ on $0 \le x \le L$ if and only if $f(0) = f(L) = 0$.

Ex.





$\Rightarrow$ $f(0) \neq 0$ $f(L) \neq 0$ $\Rightarrow$ The Fourier sine series is not continuous at $x=0$, $x=L$.