LEC6

MAY 31

Complex orthogonality.

Recall: Two real-valued functions $\psi(x)$ and $\phi(x)$ are orthogonal over $a \le x \le b$ if

$$\int_a^b \psi(x)\,\phi(x)\,dx = 0.$$

Notation: $\langle \psi, \phi \rangle = \int_a^b \psi(x)\,\phi(x)\,dx$

$\downarrow$

inner product of ψ and ϕ over $[a,b]$.

Def: Two complex valued functions ψ and ϕ are orthogonal over $a \le x \le b$ if

$$\int_a^b \overline{\psi(x)}\,\phi(x)\,dx = 0.$$

where $\overline{\psi}(x)$ is the complex conjugate:

$$\psi(x) = f(x) + i\,g(x) \qquad f(x) = \mathrm{Re}\,\psi(x) \quad \Big\} \text{ real valued.}$$
$$\Rightarrow \overline{\psi}(x) = f(x) - i\,g(x). \qquad g(x) = \mathrm{Im}\,\psi(x)$$

Note: $\overline{e^{-i\frac{n\pi x}{L}}} = e^{i\frac{n\pi x}{L}}.$

Claim: $e^{-\frac{in\pi x}{L}}$, $n = -\infty, \dots, 0, \dots, +\infty$ are orthogonal on $[-L, L]$ for different n's.

$$\int_{-L}^{L} \overline{e^{-\frac{im\pi x}{L}}}\, e^{i\frac{n\pi x}{L}}\,dx = \int_{-L}^{L} e^{i\left(\frac{m\pi x}{L} - \frac{n\pi x}{L}\right)}$$

if $m = n$, we get $2L$.

if $m \neq n$, we get

$$\int_{-L}^{L} e^{\frac{i(m-n)\pi}{L}x} = \frac{L}{i(m-n)\pi}\, e^{\frac{i(m-n)\pi x}{L}}\Bigg|_{x=-L}^{L}$$

$$= \frac{L}{i(m-n)\pi}\, 2i \cdot \sin(m-n)\pi = 0$$

<u>Note</u>: we used the fact that

$$\frac{e^{ia} - e^{-ia}}{2i} = \sin a$$

$$\therefore \int_{-L}^{L} \overline{e^{-i\frac{m\pi x}{L}}} \cdot e^{-i\frac{n\pi x}{L}}\, dx = \begin{cases} 2L & m = n \\ 0 & m \neq n. \end{cases}$$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

Use Euler's formulas
$$\begin{cases} \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} \\ \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} \end{cases}$$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \left(\frac{e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}}{2} \right) + b_n \left(\frac{e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}}{2i} \right) \right)$$

$$= a_0 + \underbrace{\frac{1}{2}\sum_{n=1}^{\infty} (a_n - ib_n)\, e^{i\frac{n\pi x}{L}}}_{\frac{1}{2}\sum_{n=-1}^{\infty} (a_{-n} - ib_{-n})\, e^{-i\frac{n\pi x}{L}}} + \frac{1}{2}\sum_{n=1}^{\infty} (a_n + ib_n)\, e^{-i\frac{n\pi x}{L}}$$

Since $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$ $\qquad a_n = a_{-n}$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$ $\qquad b_n = -b_n$

$\therefore$ define $c_0 = a_0$ $\qquad c_n = \frac{a_n + ib_n}{2}$

$\Rightarrow f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}$

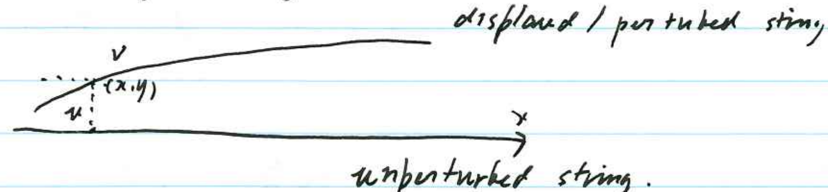
this is the <u>complex form of the Fourier series</u> of $f(x)$

<u>Note</u>: $C_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{2L}\int_{-L}^{L} f(x)\left(\cos\frac{n\pi x}{L} + i\sin\frac{n\pi x}{L}\right)dx$

$= \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{in\pi x/L}$

Ch. 4 Wave equation.

Vibrating string

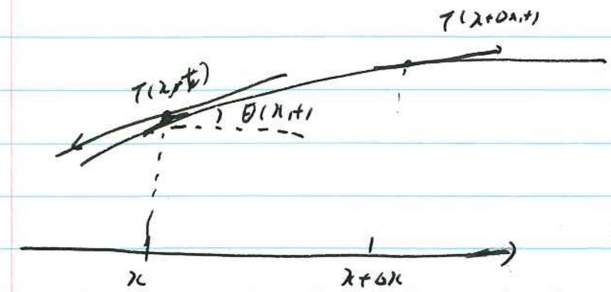


Let α be the x-coordinate of a particle / point sitting on a string. The string is in equilibrium in which it is lightly stretched.

Now perturb the string. Let u and v be vetical and horizontal displacement of the particle. Assume that the slope of the perturbed string is small $\Rightarrow$ We can neglect horizontal displacements and motion becomes completely vertical.

$\therefore\ y = u(x,t)$ where $x = \alpha$ $\qquad v \approx 0,\ x \approx \alpha$.

Goal: ~~~~ Derive equation (PDE) that governs the evolution of $u(x,t)$. We'll use the 2^{nd} Newton's law of motion : $\vec{F} = m\vec{a}$



Tensile force tries to stretch the string.

$[x, x+\Delta x]$.

small segment of the string.

Let $\rho_0(x)$ be the mass density. Then the total mass of the segment is $\rho_0(x) \cdot \Delta x$.

We assume the string is <u>perfectly flexible</u> : there is no resistance to ~~bling~~ blendings; i.e. the rest of the string exerts the force on end points that acts in the direction tangent to the string. This force is the <u>tension</u> of the string. Denote the magnitude of the tension by $T(x,t)$.

Other forces is the body force (gravity), denoted by $Q(x,t)$.

Let $\theta(x,t)$ be the angle between positive x-axis and tangent line to string.

$y = u(x,t)$ Slope of the string : $\frac{dy}{dx} = \tan\theta = \frac{\partial u}{\partial x}$

2^{nd} Newton's law of motion : $m\vec{a} = \vec{F}$

$$\rho_0(x)\,\Delta x\,\frac{\partial^2 u}{\partial t^2} = T(x+\Delta x, t)\sin\theta(x+\Delta x, t)$$

$$- T(x,t)\sin\theta(x,t) + \rho_0(x)\,\Delta x\,\underbrace{Q(x,t)}_{\text{vertical component of the body force.}}$$

Divide both sides by Δx and let $\Delta x \to 0$:

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \lim_{\Delta x \to 0} \frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)}{\Delta x}$$

$$+ \rho_0(x)\, Q(x,t).$$

$$= \frac{\partial}{\partial x}\left(T(x,t)\sin\theta(x,t)\right) + \rho_0(x)\, Q(x,t).$$

Since θ is a small angle, we have

$$\frac{\partial u}{\partial x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \approx \sin\theta \quad \text{since } \cos\theta \approx 1 \text{ for small } \theta.$$

Hence $\rho_0(x)\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial}{\partial x}\left(T(x,t)\dfrac{\partial u}{\partial x}\right) + \rho_0(x)\, Q(x,t).$

Assume the string is perfectly elastic: $T(x,t) = T_0 = \text{const}$.
(Note: In real life, most strings are nearly perfect elastic), and $T(x,t)$ depends only on local streatching.
Since $\theta(x,t)$ is small, the stretching is approximately the same as the stretching of unperturbed string, which is constant.

usually, Body force: $Q(x,t) = -g$.

Assume that the body force is neglible
$-\rho_0(x)\cdot g \ll T_0 \dfrac{\partial^2 u}{\partial x^2}$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where} \quad c^2 = \frac{T_0}{\rho_0(x)}.$$

Dimension of c: vetical length

$$\frac{V}{[t]^2} \sim c^2 \frac{V}{[L]^2} \quad \Rightarrow c^2 \sim \frac{[L]^2}{[t]^2} = \left[\frac{\text{length}}{\text{time}}\right]^2$$

$\therefore$ c has dimension of velocity.