LEC 7.

Dimension of c

$$\frac{\Delta u}{(\Delta t)^2} = c^2 \frac{\Delta u}{(\Delta x)^2} \Rightarrow C^2 = \left(\frac{\Delta x}{\Delta t}\right)^2$$

↓ velocity.

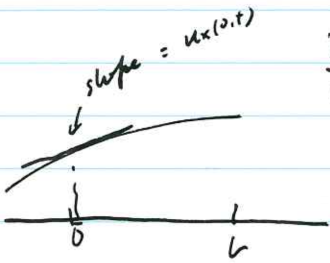
$\therefore$ c has the dimension of velocity.

In fact, $f(x-ct)$. $g(x+ct)$ are two travelling wave solutions to $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$.

Boundary condition

1. Dirichlet BCs Fix string at both ends.
   $u(0,t) = y_1(t)$ and. $u(L,t) = y_2(t)$.

2. Neumann BCs prescribe the slope of string at end points.

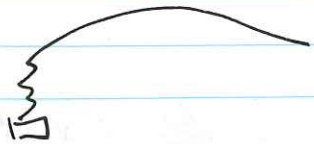


$\frac{\partial u}{\partial x}(0,t) = y_1(t)$ and

3. Elastic BC
   String is attached to mass-spring system with equilibrium position $u_E(t)$. and spring constant k.

   To $\frac{\partial u}{\partial x}(0,t) = k(u(0,t) - u_E(t))$:

   nonhomogeneous elastic BC is $u_E(t) \neq 0$.
   $u_E = 0 \Rightarrow$ homogeneous elastic BC.



derivation (see book, optional).

Remark: Elastic BC is an analogue of mixed BC for heat equations (Newton's law of cooling).

Vibrating String with Fixed ends

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & \text{on } 0<x<L \\ u(0,t) = u(L,t) = 0 & \text{BC} \\ u(x,0) = f(x) & \\ u_t(x,0) = g(x) & \end{cases} \Big\} \text{ IC.}$$

Separation of variables $u(x,t) = \phi(x)\, h(t)$.

$$\phi(x) \frac{d^2 h}{dt^2} = c^2 \frac{d^2\phi}{dx^2} h(t).$$

$$\Rightarrow \quad \frac{h''}{c^2 h} = \frac{\phi''}{\phi} = -\lambda$$

eigenvalue problem

$$\begin{cases} h'' + \lambda c^2 h = 0 \\ \phi'' + \lambda \phi = 0 \\ \phi(0) = \phi(L) = 0 \end{cases}$$

time equation $h(t)$.

$h'' + \lambda c^2 h = 0.$

$\lambda > 0.$ $\quad h(t) = C_1 \cos\sqrt{\lambda} c t + C_2 \sin\sqrt{\lambda} c t$

$\lambda = 0$ $\quad h(t) = C_1 + C_2 t$

$\lambda < 0 \quad h(t) = C_1 e^{c\sqrt{-\lambda} t} + C_2 e^{-c\sqrt{-\lambda} t}$

Physically, we expect oscillary solution in time $\Rightarrow \lambda > 0$.

<u>space equation</u>

$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(0) = \phi(L) = 0 \end{cases} \Rightarrow \text{ the only non trivial case is when } \lambda > 0.$$

$$\phi(0) = \phi(L) = 0 \quad \Rightarrow \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \dots$$

$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \cos\frac{n\pi c t}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sin\frac{n\pi c t}{L}.$$

IC: $t=0$.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \quad \rightarrow \text{ Fourier sine series for } f(x).$$

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\frac{n\pi x}{L} = g(x) \rightarrow \text{ Fourier sine series for } g(x).$$

Use the orthogonality.

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

$$B_n \cdot \frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi x}{L}\, dx$$

$$\Rightarrow B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin\frac{n\pi x}{L}\, dx.$$

<u>Note</u>: Heat equation: oscillatory in x, exponential decay in t.

Wave equation: oscillartory in both x and t.

$n=1 \Rightarrow$ fundamental frequency.

Other frequencies are multiples of the fundamental solution $u(x,t)$ is an infinite superposition of terms

normal mode of vibration : $\sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi ct}{L}+B_n\sin\frac{n\pi ct}{L}\right)$.

Trig alg.

$$A_n\cos\frac{n\pi ct}{L}+B_n\sin\frac{n\pi ct}{L}=\sqrt{A_n^2+B_n^2}\left(\overbrace{\frac{A_n}{\sqrt{A_n^2+B_n^2}}}^{\sin\theta}\cos\frac{n\pi ct}{L}+\underbrace{\frac{B_n}{\sqrt{A_n^2+B_n^2}}}_{\cos\theta}\sin\frac{n\pi ct}{L}\right)$$

(use $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\beta\sin\beta$,

$$=\sqrt{A_n^2+B_n^2}\sin\left(\frac{n\pi ct}{L}+\theta\right)$$

$\gamma=\sqrt{A_n^2+B_n^2}$: amplitude. θ : phase shift.

$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{A_n}{B_n}$.

The amplitude $\gamma=\sqrt{A_n^2+B_n^2}$ determines intensity of the sound.

$\omega=\frac{n\pi c}{L}$: # of oscillations per 2π units of time. called ~~circ~~ circular frequency.

The sound is an infinite superposition of normal mode,

$$\underbrace{\sqrt{A_n^2+B_n^2}\sin\left(\frac{n\pi ct}{L}+\theta\right)}_{A(t)}\sin\frac{n\pi x}{L} : \text{nth harmonic.}$$

$A(t)$: time-dependent amplitude.

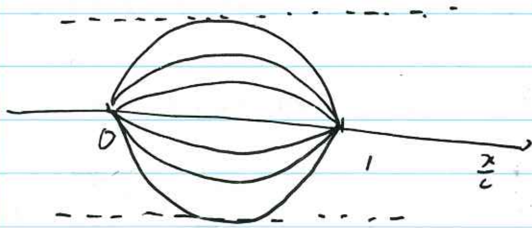
Spatial frequency is $\frac{n\pi}{L}$, while temporal frequency is $\frac{n\pi c}{L}$ which is also called natural frequency.

frequencies.

$w = \frac{\pi c}{L}$ depends on $c = \sqrt{\frac{T_0}{\rho_0}}$, L.

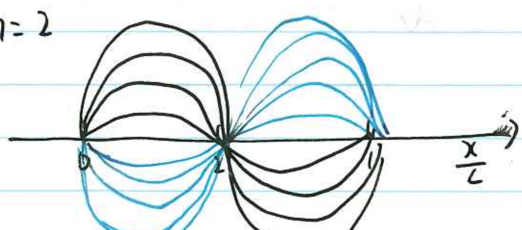
Frequency $w = \frac{\pi c}{L}$ can be increased by increasing c or decreasing L. Higher frequency gives higher pitch. Since ρ_0 is fixed, c can be changed by increasing tension T_0. Length can be damping down the string.

$n=1$



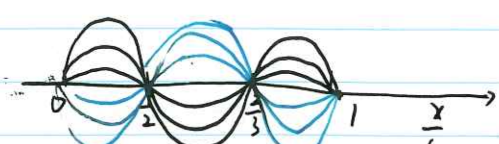
At time goes, the $\sin\left(\frac{n\pi c t}{L}+\theta\right)$ will vary between -1 and 1. Since there is no displacement at end points, we have <u>standing waves</u>.

$n=2$



Displacement is zero at $x=\frac{L}{2}$. This point is called a node.

$n=3$



3^{rd} harmonic has 2 nodes at $x=\frac{L}{3}$ and $x=\frac{2L}{3}$.

In general, nth harmonic has $(n-1)$ nodes.

Claim: $\boxed{\text{Any standing waves can be considering the sum of two travelling waves.}}$

trig. identity:

$$\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} = \frac{1}{2}\sin\frac{n\pi}{L}(x-ct) + \frac{1}{2}\sin\frac{n\pi}{L}(x+ct)$$

$$\sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L} = \frac{1}{2}\cos\frac{n\pi}{L}(x-ct) - \frac{1}{2}\cos\frac{n\pi}{L}(x+ct).$$

Hence $u(x,t)$ is a superposition of standing wave
$\Rightarrow$

$$u(x,t) = \underbrace{R(x-ct)}_{\text{wave traveling to the right with speed } c} + \underbrace{L(x+ct)}_{\text{wave traveling to the right with speed } c.}$$

Note: this is true for BC not fixed ends as well.

Recall:
$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x).$$

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi c}{L}\sin\frac{n\pi x}{L} = g(x).$$

Ex. Let $u(x,0) = f(x)$ $\quad u_t(x,0) = 0.$

$g(x) = 0 \Rightarrow B_n = 0 \quad n = 1, 2, 3 \cdots\cdots$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

$$= \sum_{n=1}^{\infty} A_n \cdot \frac{1}{2}\left[\sin\frac{n\pi}{L}(x-ct) + \sin\frac{n\pi}{L}(x+ct)\right]$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}(x-ct) + \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}(x+ct).$$

$$= \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct).$$

$$\boxed{u(x,t) = \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct)}$$