LEC 8.
JUNE 5.

Eigenvalue problems.

$$\phi'' + \lambda\phi = 0 \qquad 0 \le x \le L \text{ or } -L \le x \le L$$

with homogeneous BCs.

1. Dirichlet BC: $\phi(0) = \phi(L) = 0 \Rightarrow$ Fourier sine series.
2. Neumann BCs $\phi'(0) = \phi'(L) = 0 \Rightarrow$ Fourier cosine series.
3. Periodic BCs: $\begin{cases} \phi(-L) = \phi(L) = 0 \\ \phi'(-L) = \phi'(L) = 0 \end{cases} \Rightarrow$ Fourier series

Q: What can we say about non-constant coefficients problems?

Key concepts: linearity, homogeneity, orthogonality.

Ex. Heat flow in a non-uniformly rod.

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q$$

where $c = c(x)$, $\rho = \rho(x)$, $K_0 = K_0(x)$.

Let $Q = \alpha(x)\, u(x,t)$: heat source depends on temperature.

Then

$$c\rho u_t = (K_0 u_x)_x + \alpha(x) u \quad : \text{linear homogeneous BCs.}$$

Separation of variables. $u(x,t) = \phi(x) h(t)$.

$$c(x)\rho(x)\phi(x)\frac{dh}{dt} = h(t)\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \alpha(x)\phi(x)h(t)$$

divide by $c\rho\phi h$

$$\Rightarrow \frac{1}{h}\frac{dh}{dt} = \frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right)\cdot\frac{1}{c(x)\rho(x)\phi(x)} + \alpha(x)\frac{1}{c(x)\rho(x)} = -\lambda$$

Two ODEs: $\begin{cases} \dfrac{dh}{dt} + \lambda h = 0. \\ \dfrac{d}{dx}\left(K_0(x)\dfrac{d\phi}{dx}\right) + \alpha(x)\phi(x) + \lambda c(x)\rho(x)\phi(x) = 0 \end{cases}$

$h(t) = Ce^{-\lambda t}$

$\phi(x) = ?$ Cannot solve in general.

~~God~~ Goal. Understand behavior of $\phi(x)$ without having it in a closed form.

Ex. Circularly symmetric heat ~~fola~~ flow.
(example that physical properties are constant but equation has variable coefficients).

Recall: $\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

Circularly symmetric problem

$u = u(r,t) \Rightarrow \frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$ where $k = \frac{K_0}{c\rho}$ is a constant.

Separation of variables.

$u(r,t) = \phi(r)\, h(t)$.

$\phi(r)\frac{dh}{dt} = \frac{k}{r}\frac{d}{dr}\left(r\frac{d\phi}{dt}\right)h(t)$

→ divide by $\phi h k$ $\Rightarrow \frac{h'}{kh} = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)\cdot\frac{1}{\phi} = -\lambda$

$$\Rightarrow \begin{cases} h' + \lambda h = 0 \\ \frac{1}{r}(r\phi')' + \lambda\phi = 0. \end{cases}$$

$\Rightarrow h(t) = Ce^{-\lambda k t}$

$\phi(r) = ?$

Strum - Liouville eigenvalue problem.

Consider the ODE

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi(x) + \lambda\sigma(x)\phi(x) = 0.$$

defined on (a,b).

$\Rightarrow$ linear, homogeneous, with variable coefficients.

λ: eigenvalue.

$\phi(x) \neq 0$: associated eigenfunction.

Examples

1. $\phi'' + \lambda\phi = 0$ $\quad p=1,\ q=0,\ \sigma=1.$
2. Heat flow in a nonuniform rod.

$$\frac{d}{dx}\left(K_0(x)\frac{d\phi}{dx}\right) + \alpha(x)\phi(x) + \lambda c(x)\rho(x)\phi(x) = 0.$$

$$p(x) = K_0(x) \quad q(x) = \alpha(x) \quad \sigma(x) = c(x)\rho(x).$$

3. Circularly symmetric heat flow.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda\phi = 0 \Rightarrow \frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0$$

$$\Rightarrow p(r) = r,\ q(r) = 0,\ \sigma(r) = r.$$

Boundary conditions

	Heat eqn.	Wave eqn.	Math term.
$\phi = 0$	Fixed temperature	Fixed displacement	Dirichlet BC.
$\frac{d\phi}{dx} = 0$	Insulated.	Free ends	Neumann BC.
$\frac{d\phi}{dx} = \pm h\phi$	Newton's law of cooling	Elastic	mixed BC.
$\phi(L) = \phi(-L)$ $\phi'(L) = \phi'(-L)$	periodic	~	periodic
$\lvert\phi(0)\rvert < \infty$	bounded at the origin	~	singularity condition.

$p(x)$, $q(x)$, $\sigma(x)$ are real valued and continuous, $p(x) > 0$, $\sigma(x) > 0$ on $[a,b]$.

THM Regular Sturm-Liouville Problem.

For ~~1st first three BCs and~~ the Sturm-Liouville differential equation with regular BCs (first three BCs).

we have:

$$\begin{cases} \beta_1 \phi(a) + \beta_2 \frac{d\phi}{dx}(a) = 0 \\ \beta_3 \phi(b) + \beta_4 \frac{d\phi}{dx}(b) = 0 \end{cases}$$

①. All eigenvalues λ are real.

② $\exists$ infinitely many eigenvalues.

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$$

(a) $\exists$ a smallest eigenvalue λ_1

(b) $\nexists$ a largest eigenvalue.

③ To each eigenvalue λ_n, there exists an associated eigenfunction $\phi_n(x)$, defined up to a multiplicative constant. ($\neq 0$).

$\phi_n(x)$ has $(n-1)$ roots in (a,b).

④ The eigenfunctions $\phi_n(x)$ form a complete set in a sense that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series.

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).$$

Furthermore, this series converges to $\frac{1}{2}(f(x^+) + f(x^-))$

⑤ The eigenfunctions belongs to different eigenvalues are orthogonal to relative to the weight function $\sigma(x)$:

$$\int_a^b \phi_n(x)\, \phi_m(x)\, \sigma(x)\, dx = 0 \quad \text{if } n \neq m.$$

⑥ Any eigenvalue can be related to eigenfunction through the Raileigh Quatient

$$\lambda = \frac{-p\phi\phi'\big|_a^b + \int_a^b [p(\phi')^2 - q\phi]\, dx}{\int_a^b \phi^2 \sigma\, dx}.$$

Ex. $\begin{cases} \phi''(x) + \lambda\phi(x) = 0 \\ \phi(0) = 0 = \phi(L). \end{cases}$

$p(x) = 1$, $q(x) = 0$, $\sigma(x) = 1$. $a = 0$, $b = L$.

1. Real eigenvalues: previously, we assume the eigenvalues are real. The S-T thm proves this is true.

2. Ordering of eigenvalues.

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \cdots$$

$$\lambda_1 < \lambda_2 < \cdots.$$

$$\lambda_1 = \left(\frac{\pi}{L}\right)^2 \quad : \text{the smallest eigenvalue.}$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \to \infty \text{ as } n \to \infty.$$

3. Eigenfunctions. $\phi_n = \sin\frac{n\pi x}{L}$ $0 \le x \le L$.
in $0 < x < L$, there are exactly $n-1$ roots.

on $[0, L]$.

4. Complete set: Any piecewise smooth function on $[0,L]$ can be represented as a Fourier sine series

$$f(x) \sim \sum_{n=1}^{\infty} a_n \underbrace{\sin\frac{n\pi x}{L}}_{\phi_n(x)}.$$

5. Orthogonality. follows from orthogonality of sines.

Note: In general.

$$\int_a^b \phi_n \phi_m \sigma(x)\, dx = 0. \quad \text{if } n \ne m.$$

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).$$

$$\int_a^b f(x)\, \phi_m(x)\, \sigma(x)\, dx = \int_a^b \sum_{n=1}^{\infty} a_n \phi_n(x) \phi_m(x)\, \sigma(x)\, dx$$

$$= \sum_{n=1}^{\infty} a_n \int_a^b \phi_n(x)\, \phi_m(x)\, \sigma(x)\, dx$$

$$= a_m \int_a^b \phi_m^2(x)\,\sigma(x)\,dx.$$

$$\therefore \boxed{a_m = \frac{\int_a^b f(x)\,\phi_m(x)\,\sigma(x)\,dx}{\int_a^b \phi_m^2(x)\,\sigma(x)\,dx}}$$

~~Ap~~ $\Rightarrow \int_0^L \sin^2 \frac{m\pi x}{L}\,dx = \frac{L}{2}$

$$a_m = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx. \quad \checkmark$$

6. Rayleigh Quotient.

$$\lambda = \frac{0 + \int_0^L (\phi')^2\,dx}{\int_0^L \phi^2\,dx} = \frac{\int_0^L (\phi')^2\,dx}{\int_0^L \phi^2\,dx}$$

$(\phi')^2 \geq 0 \Rightarrow \lambda \geq 0.$

If $\lambda = 0$ $(\phi')^2 \equiv 0 \Rightarrow \phi' \equiv 0 \Rightarrow \phi(x)$ is a constant.

but $\phi(0) = \phi(L) = 0 \Rightarrow \phi(x) \equiv 0$, $\Rightarrow$ trivial solution.

$\therefore \lambda = 0$ is not an eigenvalue

Note: We showed that $\lambda > 0$ without knowing $\phi(x)$ ~~explicitly~~. explicitly.

Ex. Heat flow in a non-uniform rod.

$$\begin{cases} \frac{d}{dx}\left(K_0(x)\frac{d\phi}{dx}\right) + \lambda c(x)\rho(x)\phi(x) = 0. \\ \phi(0) = 0, \quad \frac{d\phi}{dx}(L) = 0. \end{cases} \qquad \begin{matrix} p(x) = K_0(x) > 0 \\ q(x) = 0 \end{matrix}$$

$\Rightarrow$ This is a Strum-Liouville problem. $\sigma(x) = c(x)\rho(x) > 0$.

Assume ϕ_n, λ_n can be computed.

solution: $u(x,t) = \phi_n e^{-\lambda_n t}$.

$\underset{\text{superposition}}{\Rightarrow}$ $u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t}$

IC : $u(x,0) = f(x) := \sum_{n=1}^{\infty} a_n \phi_n(x)$: generalized Fourier coefficients.

$$\Rightarrow a_n = \frac{\int_0^L f(x)\,\phi_n(x)\,c(x)\,\rho(x)\,dx}{\int_0^L \phi_n(x)\,c(x)\,\rho(x)\,dx.}$$

Q: What happens with the solution as $t \to \infty$?

Since $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdot$

λ_1 ~~be~~ will be the dominate term as t large.

$$u(x,t) \approx a_1 \phi_1(x) e^{-\lambda_1 t}.$$

Claim: $a_1 \neq 0$ and $\lambda_1 > 0$ if $f(x) > 0$.

Note: $c(x), \rho(x) > 0$.

From orthogonality.

$$a_1 = \frac{\int_0^L \overset{>0}{f(x)}\, \phi_1(x)\, \overset{>0}{c(x)}\, \overset{>0}{\rho(x)}\, dx}{\int_0^L \phi_1^2(x)\, \underset{>0}{c(x)}\, \underset{>0}{\rho(x)}} \neq 0.$$

note: $\phi_1(x)$ has the same sign since it can have $1-1=0$ zeros inside $(0,L)$. $\therefore a_1 \neq 0$.

Rayleigh quotient: $$\lambda = \frac{\cancel{-p\phi\phi'\Big|_a^b}^{\,0} + \int_a^b [p(\phi')^2 - \cancel{q\phi^2}]\,dx}{\underset{>0}{} \quad \underset{\geq 0}{} \int_a^b \phi^2 \sigma\, dx}$$

$$\Rightarrow \lambda = \frac{\int_0^L K_0(x)\,(\phi')^2\,dx}{\int_0^L \underset{\geq 0}{\phi^2}\, \underset{>0}{c(x)}\, \underset{>0}{\rho(x)}\, dx} \geq 0.$$

Check if $\lambda = 0$ is an eigenvalue.

$$\lambda = 0 \Rightarrow \int_0^L K_0(x)\,(\phi')^2 = 0$$

$$\Rightarrow \phi' \equiv 0 \Rightarrow \phi(x) \equiv \text{const}.$$

BC $\phi(0)=0 \Rightarrow \phi(x) \equiv 0$: trivial.

Hence $\lambda > 0$. In particular, $\lambda_1 > 0$.

$$u(x,t) \simeq \underbrace{a_1}_{\neq 0} \underbrace{\phi_1(x)}_{\not\equiv 0} e^{-\underbrace{\lambda_1}_{>0} t} \longrightarrow 0 \text{ as } t \to \infty.$$