LEC 9
June 6.

Higher dimension PDEs

So far, we studied. PDEs in two variables.

Heat eqn.	$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$	(x,t).
Laplace equation	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	(x,y).
Wave equation	$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$	(x,t).

Next we consider higher dimensional PDEs.

Notation: $u_x = \frac{\partial u}{\partial x}$ $\quad u_y = \frac{\partial u}{\partial y}$.

Heat eqn: $u_t = k\Delta u$ $\quad$ 2D: $u_t = k(u_{xx} + u_{yy})$

3D $\quad u_t = k(u_{xx} + u_{yy} + u_{zz})$.

Laplace eqn: $\Delta u = u_{xx} + u_{yy} + u_{zz} = 0$ $\quad$ (3D).

Wave equation:

$u_{tt} = c^2 \Delta u$ $\quad$ 2D: $u_{tt} = c^2(u_{xx} + u_{yy})$

3D: $u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz})$.

Ex. Vibrating membrane (2D)

$$\begin{cases} u_{tt} = c^2(u_{xx} + u_{yy}) & (x,y) \in \Omega. \\ u(x,y,0) = \alpha(x,y) & \\ u_t(x,y,0) = \beta(x,y) & (x,y) \in \Omega \end{cases}$$

BC: $\beta_1 u + \beta_2 \nabla u \cdot \hat{n} = 0$ on $\partial\Omega$: elastic BC

$\hat{n}$: outward unit normal. vector to $\partial\Omega$

$\beta_2 = 0 \Rightarrow$ Dirichlet BC

$\beta_1 = 0 \Rightarrow$ Neumann. BC.

Separation of variables.

$$u(x,y,t) = h(t)\,\phi(x,y).$$

$$h''\phi = c^2 h\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right).$$

$$\Rightarrow \frac{h''}{c^2 h} = \frac{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}}{\phi} = -\lambda$$

$$\begin{cases} h'' + c^2\lambda h = 0 \\ \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \lambda\phi = 0. \end{cases}$$

BC: $\beta_1\phi + \beta_2\nabla\phi\cdot\hat{n} = 0.$ on $\partial\Omega$.

Ex. Heat eqn. in 3D.

$$u_t = k(u_{xx} + u_{yy} + u_{zz}).$$

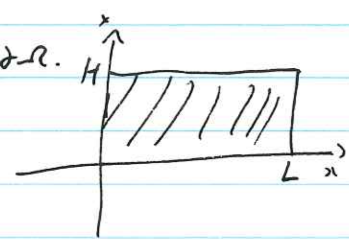
IC: $u(x,y,z,0) = \alpha(x,y,z)$, $(x,y,z)\in\Omega$

BC: $\beta_1 u + \beta_2\nabla u\cdot\hat{n} = 0$ on $\partial\Omega$.

- Separation of variables.

$$u(x,y,z,t) = h(t)\,\phi(x,y,z)$$

$$\Rightarrow \begin{cases} h' + \lambda k h = 0. \\ \phi_{xx} + \phi_{yy} + \phi_{zz} + \lambda\phi = 0. \\ \text{BC: } \beta_1\phi + \beta_2\nabla\phi\cdot\hat{n} = 0 \text{ on } \partial\Omega. \end{cases}$$



Ex. Vibrating rectangular membrane.

$$u_{tt} = c^2(u_{xx} + u_{yy}).$$

$$\Omega = \{(x,y) : 0\le x\le L,\ 0\le y\le H\}.$$

$u(x,y,z,0) = \alpha(x,y)$, $u_t(x,y,z,0) = \beta(x,y)$ $(x,y)\in\Omega$.

BC: $u=0$ on $\partial\Omega \iff u(0,y,t)=u(L,y,t)=0$

$u(x,0,t)=u(x,H,t)=0$

<u>Separation of variables.</u>

$$u(x,y,t) = \phi(x,y)\,h(t).$$

<u>time equation</u>

$h'' + \lambda c^2 h = 0 \Rightarrow \quad h(t) = c_1 \cos \sqrt{\lambda} c t + c_2 \sin(\sqrt{\lambda} c t)$

<u>space equation</u>

$$\phi_{xx} + \phi_{yy} = -\lambda \phi$$

$$\phi(0,y) = \phi(L,y) = 0 = \phi(x,0) = \phi(x,H).$$

<u>another separation of variables.</u>

$$\phi(x,y) = f(x)\,g(y).$$

$$f''g + g''f = -\lambda f g$$

$$\frac{f''}{f} + \frac{g''}{g} = -\lambda \Rightarrow \underbrace{\frac{f''}{f}}_{\text{function in } x} = \underbrace{-\frac{g''}{g} - \lambda}_{\text{function in } y.} = -\mu$$

$$\Rightarrow \begin{cases} f'' + \mu f = 0 & \text{w/ } f(0)=f(L)=0. \\ g'' + (\lambda-\mu) g = 0 & \text{w/ } g(0)=g(H)=0. \end{cases}$$

$\Rightarrow \quad f_n(x) = \sin \frac{n\pi x}{L} \qquad \mu_n = \left(\frac{n\pi}{L}\right)^2 \quad n=1,2,3,\ldots$

$\qquad g_m(y) = \sin \frac{m\pi y}{H} \qquad \lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2, \quad m=1,2,3\ldots$

$$\therefore \lambda_{nm} = \mu_n + \left(\frac{m\pi}{H}\right)^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

$n,m = 1,2,\ldots$

$\Rightarrow$ associated eigenfunctions

$$\phi_{nm}(x,y) = \sin\frac{n\pi x}{L} \cdot \sin\frac{m\pi y}{H}. \qquad n,m = 1,2,\cdots$$

By superposition,

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} \cos(\sqrt{\lambda_{mn}}\, c t)$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} \sin(\sqrt{\lambda_{mn}}\, c t)$$

Next, Let's find coefficients by using IC..

$$\alpha(x,y) = u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}$$

By orthogonality.

$$\int_0^H \alpha(x,y) \cdot \sin\frac{\tilde{m}\pi y}{H}\, dy = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sin\frac{n\pi x}{L} \int_0^H \sin\frac{m\pi y}{H} \sin\frac{\tilde{m}\pi y}{H}$$

$$\therefore \sum_{m=1}^{\infty} A_{mn} \sin\frac{n\pi x}{L} = \frac{2}{H}\int_0^H \alpha(x,y)\, \cancel{\sin\frac{m\pi y}{H}}\, \sin\frac{m\pi y}{H}\, dy.$$

Again $\frac{2}{H}\int_0^H \alpha(x,y) \sin\frac{m\pi y}{H} \sin\frac{\tilde{n}\pi x}{L}\, dx\, dy$

$$= \sum_{n=1}^{\infty} A_{mn} \int_0^L \sin\frac{n\pi x}{L} \sin\frac{\tilde{n}\pi x}{L}\, dx$$

$$\Rightarrow A_{mn} = \frac{4}{LH}\int_0^L\int_0^H \alpha(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}\, dy\, dx.$$

Similarly, $\beta(x,y) = u_t(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \underbrace{B_{mn}\sqrt{\lambda_{mn}}\, c}_{\text{coefficient.}} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}$

$$\Rightarrow B_{nm} = \frac{1}{c\sqrt{\lambda_{mn}}} \cdot \frac{4}{H}\int_0^L\int_0^H \beta(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}\, dy\, dx.$$