LEC 10

Multidimensional eigenvalue problem.

From the separation of variables to heat / wave equations, we obtained the following multidimensional eigenvalue problem.

$$\begin{cases} \Delta\phi + \lambda\phi = 0 & (x,y) \in \Omega \\ \beta_1 \phi + \beta_2 \nabla\phi \cdot \hat{n} = 0 & (x,y) \in \partial\Omega. \end{cases}$$

This is called the Helmholtz equation.
It's a special case of regular Sturm-Liouville problem in higher dimensions:

$$\nabla(p\nabla\phi) + q\phi + \lambda\sigma\phi = 0.$$

Helmholtz: $p=1$, $q=0$, $\sigma=1$.

THM. (Helmholtz equation).

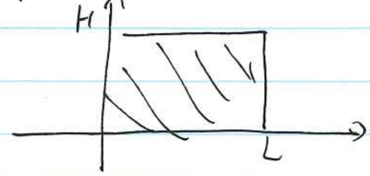
1. All eigenvalues are real.
2. $\exists$ infinitely many eigenvalues. There is the smallest one but not the largest one.
3. Corresponding to an eigenvalue, there maybe more than one eigenfunction. (Recall, in 1D S-L problem, each eigenvalue has only one eigenfunction).
4. Eigenfunctions $\phi_\lambda(x,y)$ form a complete set, namely, any piecewisely smooth function can be written as a generalized Fourier series.
   $f(x,y) = \sum_\lambda a_\lambda \phi_\lambda(x,y)$.
5. Eigenfunction corresponding to different eigenvalues are orthogonal:
   $$\iint_R \phi_{\lambda_1}\phi_{\lambda_2}\,dx\,dy = 0 \quad \text{if } \lambda_1 \neq \lambda_2.$$
   Moreover,
   different eigenfunction of the same eigenvalue

can be made orthogonal by Gram-Schmidt process, ~~that also satisfies the integral equality~~. Thus if $\lambda_1 = \lambda_2$, but ϕ_{λ_1} is linearly independent of λ_2, the same equality holds.

8. An eigenvalue λ can be related to the eigenfunction through the Rayleigh quotient.

$$\lambda = \frac{-\oint_{\partial R} \phi \nabla\phi\cdot\hat{n}\, ds + \iint_R |\nabla\phi|^2\, dx\,dy}{\iint_R \phi^2\, dx\,dy}.$$

Ex. Vibrating rectangular membrane



$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$ real.

$\lambda_{min} = \lambda_{11} = \left(\frac{\pi}{L}\right)^2 + \left(\frac{\pi}{H}\right)^2$

Multiple eigenvalues:

In general, it is possible to have more than one eigenfunction associated with the same eigenvalue.

Ex. Let $L = 2H$

$$\lambda_{nm} = \left(\frac{n\pi}{2H}\right)^2 + \left(\frac{m\pi}{H}\right)^2 = \frac{\pi}{4H^2}(n^2 + 4m^2)$$

$$\phi_{nm} = \sin\frac{n\pi x}{2H}\sin\frac{m\pi y}{H}.$$

$\left.\begin{matrix} n=4 \\ m=1 \end{matrix}\right\} \Rightarrow \lambda_{41} = \frac{20\pi^2}{4H^2} \qquad \phi_{41} = \sin\frac{4\pi x}{2H}\sin\frac{\pi y}{H}$

$\left.\begin{matrix} n=2 \\ m=2 \end{matrix}\right\} \Rightarrow \lambda_{22} = \frac{20\pi^2}{4H^2} \qquad \phi_{22} = \sin\frac{2\pi x}{2H}\sin\frac{2\pi y}{H}$

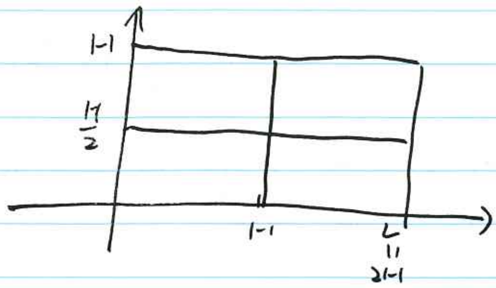
Nodal curves: $\phi = 0$.

Ex. $n=2$ $m=2$. $\phi_{22} = \sin\frac{\pi x}{H}\sin\frac{2\pi y}{H}$

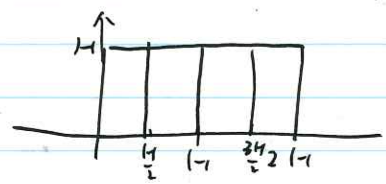
$\phi_{22} = 0 \Rightarrow \sin\frac{\pi x}{H} = 0$ or $\sin\frac{2\pi y}{H} = 0$

$$\sin\frac{\pi x}{H}=0 \Rightarrow \frac{\pi x}{H}=k\pi \Rightarrow x=kH \quad \therefore\ x=0,\ H,\ 2H$$

$$\sin\frac{2\pi y}{H}=0 \Rightarrow \frac{2\pi y}{H}=l\pi \Rightarrow y=\frac{lH}{2} \quad y=0,\ \frac{H}{2},\ H$$



Ex. $n=4$ $m=1$ $\quad \phi_{41}(x,y)=\sin\frac{2\pi x}{H}\sin\frac{\pi y}{H}$



Note: Nodal curves of ϕ_{22} and ϕ_{41} are different $\Rightarrow$ ϕ_{22}, ϕ_{41} are different.

4. Series of eigenfunctions.

Any piecewise smooth functions can be written as

$$f(x,y) \sim \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\phi_{nm}(x,y)$$

Convergence:

mean - square error

$$E \equiv \iint_R \left(f-\sum_{\lambda}^{M} a_\lambda \phi_\lambda\right)^2 dx\,dy,$$

as M increases, $E \to 0$, i.e. the truncated generalized Fourier series converges to f in the mean square sense.

3. Orthogonality of eigenfunctions.

$$f(x,y) \sim \sum_\lambda a_\lambda \phi_\lambda(x,y).$$

$$\Rightarrow \iint_\Omega f \phi_{\lambda i}\, dxdy = \sum_\lambda a_\lambda \iint_\Omega \phi_\lambda \phi_{\lambda i}\, dxdy.$$

by orthogonality, $\iint_\Omega \phi_\lambda \phi_{\lambda i}\, dxdy = 0$ if $\lambda_i \neq \lambda$.

$$\Rightarrow a_i = \frac{\iint_\Omega f \phi_{\lambda i}\, dxdy}{\iint_\Omega \phi_{\lambda i}^2\, dxdy}.$$

Special case. (rectangular membrane)

$$a_{nm} = \frac{\int_0^H \int_0^L f(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}\, dxdy}{\int_0^H \int_0^L \sin^2\frac{n\pi x}{L} \sin^2\frac{m\pi y}{H}\, dxdy}$$

$$= \frac{4}{HL} \int_0^H \int_0^L f(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}\, dxdy.$$

Green's formula.

Recall Helmholtz equation.

$$\Delta\phi + \lambda\phi = 0 \quad w/ \quad \beta_1\phi + \beta_2 \nabla\phi\cdot\hat{n} = 0 \text{ on } \partial\Omega.$$

Let $L = \Delta$: Laplace operator.

$$L\phi + \lambda\phi = 0.$$

Let u, v be two functions in x, y.

$$uL(v) - vL(u) = u\Delta v - v\Delta u.$$

Note: $\nabla\cdot(u\nabla v) = \nabla u\cdot\nabla v + u\nabla\cdot\nabla v = \nabla u\cdot\nabla v + u\Delta v$

$\nabla\cdot(v\nabla u) = \nabla v\cdot\nabla u + v\Delta u.$

$$\Rightarrow \quad \boxed{u\Delta v - v\Delta u = \nabla \cdot (u\nabla v - v\nabla u)}$$

This is called the Lagrange identity.

Integration on Ω.

$$\iint_\Omega (u\Delta v - v\Delta u)\,dx\,dy = \iint_\Omega \nabla \cdot (u\nabla v - v\nabla u)\,dx\,dy$$

By divergence thm: $\iint_\Omega \nabla \cdot \vec{A}\,dx\,dy = \int_{\partial\Omega} \vec{A}\cdot\hat{n}\,ds$

$$\boxed{\iint_\Omega (u\Delta v - v\Delta u)\,dx\,dy = \oint_{\partial\Omega} (u\nabla v - v\nabla u)\cdot\hat{n}\,dS.}$$

This is called the <u>Green's formula.</u>

<u>Def:</u> The operator $L = \Delta$ is <u>self-adjoint</u> in the following sense.

If u, v are two functions such that

$$\oint_{\partial\Omega} (u\nabla v - v\nabla u)\hat{n}\,dS = 0 \qquad (*).$$

then $\iint_\Omega (u\Delta v - v\Delta u)\,dx\,dy = 0.$

<u>Note</u>: $(*)$ is satisfied by BC: $\beta_1\phi + \beta_2\nabla\phi\cdot\hat{n} = 0$ on $\partial\Omega$.

1. $\beta_2 = 0 \Rightarrow \phi|_{\partial\Omega} = 0 \quad \Rightarrow \quad u|_{\partial\Omega} = v|_{\partial\Omega} = 0.$ ✓

2. $\overset{\beta_1=0}{\cancel{\beta_2}} \Rightarrow \nabla u\cdot\hat{n} = 0, \quad \nabla v\cdot\hat{n} = 0$ on $\partial\Omega$. ✓

3. $\beta_1 u + \beta_2\nabla u\cdot\hat{n} = 0 \qquad \beta_1 v + \beta_2\nabla v\cdot\hat{n} = 0.$

$$\oint_{\partial\Omega} [u\underbrace{\nabla v\cdot\hat{n}}_{-\frac{\beta_1}{\beta_2}v} - v\underbrace{\nabla u\cdot\hat{n}}_{-\frac{\beta_1}{\beta_2}u}]\,dS = \oint_{\partial\Omega} [uv(-\tfrac{\beta_1}{\beta_2}) + uv(\tfrac{\beta_1}{\beta_2})]$$

$$= 0 \quad ✓$$

Claim: Any two eigenfunctions corresponding to different eigenvalues are orthogonal.

Pf: 1) $\Delta \phi_a + \lambda_a \phi_a = 0$

2) $\Delta \phi_b + \lambda_b \phi_b = 0$

$$\underset{\uparrow \atop \text{self-adjointness}}{0} = \iint_\Omega \left[\phi_a \overbrace{\Delta \phi_b}^{-\lambda_b \phi_b} - \phi_b \overbrace{\Delta \phi_a}^{-\lambda_a \phi_a} \right] dx\,dy$$

$$= \iint_\Omega \underbrace{(\lambda_a - \lambda_b)}_{\neq 0} \phi_a \phi_b \, dx\,dy = (\lambda_a - \lambda_b) \iint_\Omega \phi_a \phi_b \, dx\,dy$$

$$\Rightarrow \iint_\Omega \phi_a \phi_b \, dx\,dy = 0.$$

Claim: Eigenvalues of Helmholtz equation are real.

Pf: Sps λ is complex w/ complex conjugate $\bar{\lambda}$.

$\Delta \phi + \lambda \phi = 0 \Rightarrow \Delta \bar{\phi} + \bar{\lambda} \bar{\phi} = 0 \Rightarrow \bar{\lambda}$ is an eigenvalue w/ associated eigenfunction $\bar{\phi}$.

By orthogonality, $(\lambda - \bar{\lambda}) \iint_\Omega \phi \bar{\phi} \, dx\,dy = 0$.

write $\phi = \underbrace{\phi_r + i \phi_i}_{\text{real valued}} \Rightarrow \phi \bar{\phi} = (\phi_r + i\phi_i)(\phi_r - i\phi_i)$

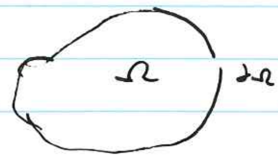
$= \phi_r^2 + \phi_i^2 > 0$ since $\phi \neq 0$.

Hence $(\lambda - \bar{\lambda}) = 0. \Rightarrow \lambda$ is real..

Optional reading: Gram - Schmidt process (appendix).

Rayleigh Quotient.

$\Delta\phi + \lambda\phi = 0.$



multiply both side by ϕ and integrate over Ω.

$$\iint_\Omega \phi\Delta\phi + \lambda\phi^2 \, dxdy = 0$$

$$\Rightarrow \quad \lambda = - \frac{\iint_\Omega \phi\Delta\phi \, dxdy}{\iint_\Omega \phi^2 \, dxdy}.$$

Note: $\nabla\cdot(f\vec{g}) = \cancel{\nabla f\cdot g + f} \; \nabla f\cdot g + f\nabla\cdot\vec{g}$

Let $f = \phi \quad \vec{g} = \nabla\phi$

$$\nabla\cdot(\phi\nabla\phi) = \underbrace{\nabla\phi\cdot\nabla\phi}_{|\nabla\phi|^2} + \phi\Delta\phi$$

$$\therefore \quad \phi\Delta\phi = \nabla\cdot(\phi\nabla\phi) - |\nabla\phi|^2.$$

$$\therefore \quad \lambda = \frac{-\iint_\Omega \nabla\cdot(\phi\nabla\phi) \, dxdy + \iint_\Omega |\nabla\phi|^2 \, dxdy}{\iint_\Omega \phi^2 \, dxdy}.$$

$$= \frac{-\oint_{\partial\Omega} \phi\nabla\phi\cdot\vec{n} \, ds + \iint_\Omega |\nabla\phi|^2 \, dxdy}{\iint_\Omega \phi^2 \, dxdy}.$$

Ex. Heat eqn.

$u_t = k\Delta u \quad w/ \quad u|_{\partial\Omega} = 0.$

$u(x,y,t) = e^{-\lambda kt}\phi(x,y)$

BVP $\quad \Delta\phi + \lambda\phi = 0 \quad w/ \quad \phi|_{\partial\Omega} = 0.$

Rayleigh quotient.

$$\lambda = \frac{\iint_\Omega |\nabla \phi|^2 dx\,dy}{\iint_\Omega \phi^2 dx\,dy} \geq 0.$$

$\lambda = 0 \Rightarrow |\nabla \phi| = 0 \Rightarrow \phi = \text{const} = 0$ since $\phi|_{\partial\Omega} = 0$.

$\therefore \lambda > 0.$

Moreover, $\lim_{t\to\infty} u(x,y,t) = 0.$

Ex. $u_t = k \Delta u$ w/ $\nabla u \cdot \hat{n} = 0$ on $\partial\Omega$.

$$\lambda = \frac{\iint_\Omega |\nabla \phi|^2 dx\,dy}{\iint_\Omega \phi^2 dx\,dy} \geq 0 \Rightarrow \lambda \geq 0.$$

~~same as before~~. $\lambda = 0 \Rightarrow \phi \equiv \text{const}.$

$\therefore \lim_{t\to\infty} u(x,y,t) = u_\infty \equiv \text{const}.$

From integral conservation law.

$$\frac{d}{dt} \iint_\Omega c\rho u \, dx\,dy = \oint_{\partial\Omega} K_0 \nabla u \cdot \hat{n} \, ds = 0.$$

$$\therefore \iint_\Omega c\rho u \equiv \text{const}.$$

$\cancel{\iint_\Omega c\rho u(x,y}$

$$\iint_\Omega c\rho u(x,y,0)\,dx\,dy = \iint_\Omega c\rho \underbrace{u_\infty}_{\text{const}} \,dx\,dy$$

$$\therefore \quad u_\infty = \frac{\iint_\Omega f(x,y)\,dx\,dy}{\iint_\Omega dx\,dy} \Rightarrow \text{average of initial temperature.}$$