LEC 13 Nonhomogeneous problems

General problem

Heat eqn. $u_t = u_{xx} + Q(x,t)$

$u(0,t) = A(t)$ $\quad u(L,t) = B(t)$ $\quad u(x,0) = f(x)$.

Wave eqn. $u_{tt} = u_{xx} + Q(x,t)$.

$u(0,t) = A(t)$ $\quad u(L,t) = B(t)$ $\quad u(x,0) = f(x)$ $\quad u_t(x,0) = g(x)$.

Poisson equation. $u_{xx} + u_{yy} = Q(x,y)$

$u(x,0) = f_1(x)$ $\quad u(x,H) = f_2(x)$.

$u(0,y) = g_1(y)$ $\quad u(L,y) = g_2(y)$.

Time independent BCs, no source

$$\begin{cases} u_t = k u_{xx} \quad w/ \quad u(0,t) = A \quad u(L,t) = B \\ u(x,0) = f(x). \end{cases}$$

Equilibrium temp: $\lim_{t\to\infty} u(x,t) = w(x)$.

$$\begin{cases} w'' = 0 \\ w(0) = A \cdot \quad w(L) = B \end{cases} \Rightarrow w = A + \frac{B-A}{L} x$$

Displacement from equilibrium $\quad u(x,t) = v(x,t) + w(x)$

$v_t = k v_{xx}$ $\quad w/ \quad$ $v(0,t) = v(L,t) = 0$.

$v(x,0) = f(x) - [A + \frac{B-A}{L} x] \equiv g(x)$.

$$\Rightarrow v(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t}$$

w/ $\quad a_n = \frac{2}{L}\int_0^L g(x)\cdot \sin\frac{n\pi x}{L}\,dx$

II. Steady source.

$u_t = k u_{xx} + Q(x) \quad$ w/ $\quad u(0,t) = A \quad u(L,t) = B.$

$u(x,0) = f(x)$

Equilibrium temp. $\quad w'' = -\frac{1}{k}Q(x) \quad w(0) = A,\ w(L) = B$

$$\Rightarrow w' = C_1 - \frac{1}{k}\int_0^x Q(\xi)\,d\xi$$

$$w(x) = C_1 x + C_2 - \frac{1}{k}\int_0^x \int_0^\xi Q(\eta)\,d\eta\,d\xi$$

$$w(0) = A \Rightarrow C_2 = A$$

$$w(L) = B = C_1 L + C_2 + \frac{1}{k}\int_0^L\int_0^\xi Q(\eta)\,d\eta\,d\xi$$

$$\therefore w(x) = \frac{x}{L}\left[B - A + \frac{1}{k}\int_0^L\int_0^\xi Q(\eta)\,d\eta\,d\xi\right] + A - \frac{1}{k}\int_0^x\int_0^\xi Q(\eta)\,d\eta\,d\xi.$$

$v_t = k v_{xx} \quad$ w/ $\quad v(0,t) = v(L,t) = 0.$

$v(x,0) = f(x) - w(x) = g(x).$

General case.

$u_t = k u_{xx} + Q(x,t) \qquad u(x,0) = f(x).$

$u(0,t) = A(t) \qquad u(L,t) = B(t).$

In general, equilibrium may not exist.

Can always make BC homogeneous by defining any reference temp. distribution, $r(x,t)$ that satisfies the BCs: $\quad r(0,t) = A(t), \quad r(L,t) = B(t)$

simplest choice: $r(x,t) = A(t) + \frac{x}{L}[B(t) - A(t)]$.

Let $u(x,t) = v(x,t) + r(x,t)$.

$$v_t = k v_{xx} + \bar{Q}(x,t) \qquad \cancel{r(x,t)} \qquad v(x,0) = f(x) - r(x,0)$$
$$v(0,t) = v(L,t) = 0.$$

where $\bar{Q}(x,t) = Q(x,t) - r_t + k\, r_{xx}$.

Eigenfunction expansion.

Works for general source terms w/ homogeneous BCs.

$$v_t = L(v) + \bar{Q}(x,t). \qquad v(x,0) = g(x).$$
$$v(0,t) = v(L,t) = 0 \qquad (L \text{ contains only partial derivatives}).$$

Step 1.: solve problem w/ $\bar{Q}(x,t) = 0$

$$v(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x) \quad \text{where} \quad \begin{cases} L(\phi_n) + \lambda_n \phi_n = 0 \\ \phi(0) = \phi(L) = 0. \end{cases}$$

Step 2: Let $a_n = a_n(t)$ and plug ~~in~~ back into full equation

$L = \frac{\partial^2}{\partial x^2}$

$$v(x,t) = \cancel{\sum a_n \phi_n} \sum_{n=1}^{\infty} a_n(t)\, \phi_n(x) \qquad v_t = \sum_{n=1}^{\infty} a_n'(t)\, \phi_n(x).$$

$$L(v) = \sum_{n=1}^{\infty} a_n(t)\, L(\phi_n(x)) = -\sum_{n=1}^{\infty} a_n(t)\, \lambda_n\, \phi_n(x) k.$$

$$\text{Let } \bar{Q}(x,t) = \sum_{n=1}^{\infty} \bar{q}_n(t)\, \phi_n(x).$$

$$\bar{q}_n(t) = \frac{\int_0^L \bar{Q}(x,t)\, \phi_n(x)\, dx}{\int_0^L \phi_n^2(x)\, dx}$$

$$\Rightarrow \sum_{n=1}^{\infty} [\, a_n'(t) + k\lambda_n a_n(t) - \bar{q}_n(t) \,] \phi_n(x) = 0.$$

$$\therefore \quad a_n' + k\lambda_n a_n = \bar{q}_n \quad : \text{ODEs for coefficients.}$$

@ $t=0$ $\quad v(x,0) = g(x) = \sum_{n=1}^{\infty} a_n(0)\, \phi_n(x).$

$$\therefore \quad a_n(0) = \frac{\int_0^L g(x)\, \phi_n(x)\, dx}{\int_0^L \phi_n^2(x)\, dx}$$

Using the integrating factor.

$$\underbrace{e^{k\lambda_n t} a_n' + k\lambda_n e^{k\lambda_n t} a_n}_{= [a_n e^{k\lambda_n t}]'} = e^{k\lambda_n t} \bar{q}_n$$

$$\Rightarrow \quad a_n(t) e^{k\lambda_n t} - a_n(0) = \int_0^t e^{k\lambda_n \tau} \bar{q}_n(\tau)\, d\tau$$

$$\therefore \quad a_n(t) = e^{-k\lambda_n t} a_n(0) + e^{-k\lambda_n t} \int_0^t e^{k\lambda_n \tau} \bar{q}_n(\tau)\, d\tau.$$

<u>Note</u>: PDE, IC are satisfied.
BC are satisfied due to ~~BCs~~ ϕ_n.