LEC 14. **Ex.** $u_t = u_{xx} + \sin(3x)e^{-t}$, $u(x,0) = f(x)$

$u(0,t) = 0.$

$u(\pi,t) = 1.$

First, make BC homogeneous $\quad v(x,t) = u(x,t) - \frac{x}{\pi}$

$$\begin{cases} v_t = v_{xx} + \sin(3x)e^{-t} & v(0) = f(x) - \frac{x}{\pi} \\ v(0,t) = v(\pi,t) = 0 \end{cases}$$

$$\Rightarrow v(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin(nx) \qquad \lambda_n = n^2$$

$$v(x,0) = f(x) - \frac{x}{\pi} = \sum_{n=1}^{\infty} a_n(0) \sin nx$$

$$\Rightarrow a_n(0) = \frac{2}{\pi}\int_0^{\pi} \left(f(x) - \frac{x}{\pi}\right) \sin nx \, dx$$

$$\bar{Q}(x,t) = \sin 3x \, e^{-t} = \sum_{n=1}^{\infty} \bar{q}_n(t)\, \phi_n(x) = \sum_{n=1}^{\infty} \bar{q}_n(t) \sin(nx)$$

$$\bar{q}_n(t) = \frac{2}{\pi}\int_0^{\pi} e^{-t} \sin(3x) \sin nx \, dx = \begin{cases} 0 & \text{if } n \neq 3 \\ e^{-t} & \text{if } n = 3. \end{cases}$$

$$\therefore a_n(t) = a_n(0) e^{-n^2 t} + e^{-n^2 t} \int_0^t \bar{q}_n(\tau) e^{n^2 \tau} \, d\tau.$$

$$a_n(t) = \begin{cases} a_n(0) e^{-n^2 t} & \text{if } n \neq 3 \\ a_3(0) e^{-9t} + e^{-9t} \int_0^t e^{-\tau} e^{9\tau} \, d\tau. \end{cases}$$

More on source terms.

Consider the heat eqn w/ nonhomogeneous BCs.

$$\begin{cases} u_t = k u_{xx} + Q(x,t) & u(x,0) = f(x) \\ u(0,t) = A(t) & u(L,t) = B(t) \end{cases}$$

If we ignore the source term and pretend the BCs are homogeneous, then the separation of variables yields.

$$\begin{cases} \phi_n'' + \lambda_n \phi_n = 0 \\ \phi_n(0) = \phi_n(L) = 0. \end{cases}$$

assume $\quad u(x,t) = \sum_{n=1}^{\infty} b_n(t)\, \phi_n(x).$

Note: Technically we should write $u(x,t) \sim \sum_{n=1}^{\infty} b_n(t)\, \phi_n(x)$ sim at $x=0$ and $x=L$ the series converges to 0, rather than $A(t)$ and $B(t)$.

$$\Rightarrow u_t = \sum_{n=1}^{\infty} b_n'(t)\, \phi_n(x).$$

Let $Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\, \phi_n(x)$. where

$$q_n(t) = \frac{\int_0^L Q(x,t)\, \phi_n(x)\, dx}{\int_0^L \phi_n^2(x)\, dx}$$

$$u_t = k u_{xx} + Q(x,t).$$

$$\Rightarrow \sum_{n=1}^{\infty} b_n'(t)\, \phi_n(x) = k u_{xx} + \sum_{n=1}^{\infty} q_n(t)\, \phi_n(x)$$

Orthogonality and linear independence $\Rightarrow$

$$b_n'(t) = \frac{\int_0^L [k u_{xx}]\, \phi_n(x)\, dx}{\int_0^L \phi_n^2\, dx} + q_n(t).$$

$$\int_0^L u_{xx}\,\phi_n(x)\,dx \xlongequal{\text{int. by parts.}} \overset{\overset{0}{\shortparallel}}{\phi_n(x)\,u_x\Big|_0^L} - \int_0^L \phi_n'\,u_x\,dx$$

$$= -\int_0^L \phi_n'\,u_x\,dx \xlongequal{\text{I.B.P.}} -\left[u\,\phi_n'\right]\Big|_0^L + \int_0^L u\,\underbrace{\phi_n''}_{-\lambda_n\phi_n}\,dx$$

Since $\phi_n(x) = \sin\frac{n\pi x}{L} \Rightarrow \phi_n'(x) = \frac{n\pi}{L}\cos\frac{n\pi x}{L}$

$\therefore$ $\left[A(t) - (-1)^n B(t)\right]\cdot\frac{n\pi}{L} - \lambda_n\int_0^L u(x,t)\,\phi_n(x)\,dx$

Note: $b_n(t) = \dfrac{\int_0^L u(x,t)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}$

$$\therefore\quad b_n'(t) + k\,\lambda_n b_n(t) = q_n(t) + \frac{k\left(\frac{n\pi}{L}\right)\left[A(t) - (-1)^n B(t)\right]}{\int_0^L \phi_n^2(x)\,dx}.$$

$$\text{w/}\quad b_n(0) = \frac{\int_0^L f(x)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2\,dx}.$$

Ex. $u_t = u_{xx} + \sin(3x)e^{-t}$ $\quad u(0,t) = 0,\ u(\pi,t) = 1.$

We showed $u(x,t) = \frac{x}{\pi} + (e^{-t} - e^{9t}) + \sum_{n=1}^{\infty} a_n(0)\sin nx\, e^{-n^2 t}$

New approach.

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin nx$$

$$b_n' + n^2 b_n = q_n + 2n\left[0 - (-1)^n\right]$$

$$q_n = \begin{cases} 0 & \text{if } n \neq 3 \\ e^{-t} & \text{if } n = 3. \end{cases}$$

multiply by $e^{n^2 t}$. (Integrating factor)

$$[e^{n^2 t} b_n]' = \begin{cases} e^{(n^2-1)t} - 2n(-1)^n e^{n^2 t} & n=3 \\ -2n(-1)^n e^{n^2 t} & n \neq 3. \end{cases}$$

Integrate. ~~and~~ we can get all b_n's.

Which method is better?
In general, the first approach converges faster.

<u>Poisson's equation</u> (Laplace equation w/ source).

$$\begin{cases} \Delta u = Q & \text{on } \Omega \\ u = \alpha & \text{on } \partial\Omega. \end{cases}$$

Problem is nonhomogeneous:

1. nonhomogeneous BC
2. source term.

Let $u = u_1 + u_2$.

Problem #1. $\begin{cases} \Delta u_1 = Q \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases}$

Problem #2 $\begin{cases} \Delta u_2 = 0 \\ u_2 = \alpha & \text{on } \partial\Omega. \end{cases}$

We already know how to solve #2. (by sep. of variables).
(r.f. Ch.2).