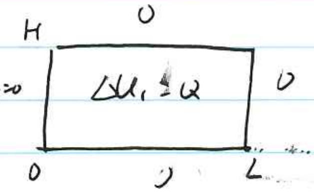
LEC 15

Let's look at problem 1.



approach #1. 1D eigen fn.

$$\begin{cases} \Delta u_1 = 0 & \text{in } \Omega. \\ u_1 = 0 & \text{on } \partial\Omega \end{cases}$$

Take 1D eigenfunctions that satisfy BC, (can choose either $\sin\frac{n\pi x}{L}$ or ~~$\sin\frac{n\pi x}{}$~~ $\sin\frac{n\pi y}{H}$

Let $u_1(x,y) = \sum_{n=1}^{\infty} b_n(y) \sin\frac{n\pi x}{L}$ or $u(x,y) = \sum_{n=1}^{\infty} C_n(x) \sin\frac{m\pi y}{H}$

$$\Delta u_1 = \sum_{n=1}^{\infty} \left[b_n''(y) - \left(\frac{n\pi}{L}\right)^2 b_n(y) \right] \sin\frac{n\pi x}{L} = Q(x,y).$$

Let $Q(x,y) := \sum_{n=1}^{\infty} q_n(y) \sin\frac{n\pi x}{L}$

Now $b_n'' - \left(\frac{n\pi}{L}\right)^2 b_n = q_n$ w/ $b_n(0) = b_n(H) = 0$.

This is a 2nd order ODE, nonhomogeneous.
Can solve by variation of parameters.

$$b_n(y) = \sinh\left(\frac{n\pi(H-y)}{H}\right) \int_0^y q_n(\xi) \sinh\left(\frac{n\pi\xi}{L}\right) d\xi +$$

$$\sinh\left(\frac{n\pi y}{L}\right) \int_y^H q_n(\xi) \sinh\left(\frac{n\pi(H-\xi)}{L}\right) d\xi$$

Approaches #2. 2D eigenfunction.

From sec 7.3 $\quad \phi_{nm}(x,y) = \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}$

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2.$$

Let $u_1(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}$

$$\Delta u_1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{nm}\,\Delta\phi_{nm} = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \lambda_{nm} b_{nm} \phi_{nm} = 0.$$

By orthogonality

$$b_{nm} = -\frac{1}{\lambda_{nm}}\frac{4}{LH}\int_0^H\int_0^L Q(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\,dx\,dy$$

Note: ① Approaches #1 converges faster since it's only a single sum. rather than a double sum.

② In general, #2 is easier to work with since it only requires to apply the orthogonality. Approache #1 requires the solution of non-hom. ODEs.

Another approach to nonhomogeneous BCs

$$\Delta u = Q \qquad u = \alpha \text{ on } \partial\Omega.$$

Step 1: solve $\Delta\phi = -\lambda\phi$ w/ $\phi = 0$ on $\partial\Omega$.

Step 2: Let $u(x,y) = \sum_n\sum_m b_{nm}\phi_{nm}(x,y)$

Orthogonality: $b_{nm} = -\frac{1}{\lambda_{nm}}\dfrac{\iint u\,\Delta\phi_{nm}\,dx\,dy}{\iint \phi_{nm}^2\,dx\,dy}$.

Integration by parts

$$\iint u\,\Delta\phi_{nm}\,dx\,dy = \iint \phi_{nm}\,\Delta u + \oint_{\partial\Omega}(u\nabla\phi_{nm} - \phi_{nm}\nabla u)\cdot\hat{n}\,ds$$

Note $\Delta u = Q$

$$\therefore \quad b_n = -\frac{1}{\lambda_n} \frac{\iint_\Omega \phi_{nm} Q \, dx\,dy + \oint_{\partial\Omega} \alpha \nabla \phi_{nm} \cdot \hat{n} \, ds}{\iint_\Omega \phi_{nm}^2 \, dx\,dy}.$$

Fourier transform

Heat equation on infinite domain

$$\begin{cases} \dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2} & -\infty < x < \infty \\ \text{IC:} \quad u(x,0) = f(x). \end{cases}$$

BCs? Simplest case, suppose 0 at infinity, i.e.
$u(-\infty, t) = 0 = u(\infty, t)$.

Separation of variables.

$$\begin{cases} h' = -\lambda k h \\ \phi'' = -\lambda \phi \end{cases}$$

no λ s.t. ϕ can approach 0 at both $\pm\infty$.

Modify the BC $\Rightarrow$ bounded at infinity.

Eigenvalue problem:

$$\begin{cases} \phi'' + \lambda \phi = 0 \\ |\phi(\pm\infty)| < \infty. \end{cases}$$

if $\lambda < 0$, ϕ has exponential growth, hence cannot be bounded.

If $\lambda > 0$. $\phi = c_1 \cos\sqrt{\lambda} x + c_2 \sin\sqrt{\lambda} x$

$\lambda = 0 \Rightarrow \phi \equiv \text{const}$ and bounded at $\pm\infty$.

For Fourier series, λ is <u>discrete</u> (since $\lambda = (\frac{n\pi}{L})^2$)

Now λ is ~~continuous~~ continuous and we call λ the continuum spectrum.

<u>Superposition principle.</u>

<u>product sol:</u> $\sin\sqrt{\lambda}x\, e^{-\lambda kt}$ and $\cos\sqrt{\lambda}x\, e^{-\lambda kt}$

summing over all $\lambda \geq 0$. (In fact, integrate).

$$u(x,t) = \int_0^\infty [C_1(\lambda)\cos\sqrt{\lambda}x\, e^{-\lambda kt} + C_2(\lambda)\sin\sqrt{\lambda}x\, e^{-\lambda kt}]\, d\lambda$$

Usually we let $\lambda = w^2$

$$u(x,t) = \int_0^\infty A(w)\cos wx\, e^{-kw^2t} + B(w)\sin wx\, e^{-kw^2t}\, dw$$

Note previous, we solve the heat equation on finite domain with periodic BCs.

$$u(x,t) = \sum_{n=0}^\infty \left(A_n \cos\frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t}\right) + \sum_{n=1}^\infty B_n \sin\frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t}$$

For IC, $u(x,0) = f(x)$, we have

$$f(x) = \int_0^\infty A(w)\cos wx + B(w)\sin wx\, dx.$$

Now we can use e^{-iwx} replace $\sin wx$, $\cos wx$

w/ w ranges from ~~to~~ $0, \pm 1, \pm 2, \cdots$

then $u(x,t) = \int_{-\infty}^\infty C(w)\, e^{-iwx} e^{-kw^2t}\, dw$

w/ $f(x) = \int_{-\infty}^\infty C(w)\, e^{-iwx}\, dw$

☆ Note: In most literatures, we pick $e^{i\omega x}$ here rather than $e^{-i\omega x}$. But we need to stick with the convention of the book! :(.

We shall derive a formula for $c(\omega)$ shortly and ~~[illegible]~~ evaluate the integral of $u(x,t)$.

Fourier transform pair

Recall the complex form of Fourier series.

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{in\pi x/L}\, dx.$$

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}$$

The Fourier series identity

$$f(x) \sim \sum_{n=-\infty}^{\infty} \left[\frac{1}{2L}\int_{-L}^{L} f(y)\, e^{in\pi y/L}\, dy\right] e^{-in\pi x/L}$$

For Fourier transform, $L \to \infty$.

$$f(x) \sim \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x)\, e^{i\omega y}\, dy\right] e^{-i\omega x}\, d\omega$$

We then define $F(\omega)$ the Fourier transform of $f(x)$ to be

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx$$

sometime, write $F(\omega) = \hat{f}(\omega)$.

It follows that, $f(x) \sim \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-i\omega x}\, d\omega$.

~~Exam~~ Inverse Fourier transform of a Gaussian.

In order to compute the solution of the heat equation, we need to compute the inverse Fourier transform of the Gaussian. Like $G(w) = e^{-\beta w^2}$.

In fact $g(x) = \int_{-\infty}^{\infty} G(w)\, e^{-iwx}\, dw = \int_{-\infty}^{\infty} e^{-\beta w^2} e^{-iwx}\, dx$

$$= \sqrt{\frac{\pi}{\beta}}\, e^{-x^2/4\beta}$$

Also, if $f(x) = e^{-\alpha x^2}$

$$\hat{f} = \frac{1}{\sqrt{4\pi\alpha}}\, e^{-w^2/4\alpha}.$$

Remark: By above formulas, we see that the (inverse) Fourier transform of a Gaussian is still a Gaussian.