LEC 16.

Gaussian : $e^{-\beta w^2}$

Fourier transform of a Gaussian.

$$g(x) = \int_{-\infty}^{\infty} e^{-\beta w^2} e^{-iwx}\, dx$$

How should we evaluate it?

$$g'(x) = \int_{-\infty}^{\infty} -iw e^{-\beta w^2} e^{-iwx}\, dw$$

Using integration by parts

$$g'(x) = \frac{i}{2\beta} \int_{-\infty}^{\infty} \frac{d}{dw}(e^{-\beta w^2})\, e^{-iwx}\, dw$$

$$= -\frac{x}{2\beta} \int_{-\infty}^{\infty} e^{-\beta w^2} e^{-iwx}\, dw = -\frac{x}{2\beta} g(x).$$

$\therefore$ We have the ODE $\quad g' = -\frac{x}{2\beta} g.$

The solution is $\quad g(x) = g(0)\, e^{-x^2/4\beta}$

with $\quad g(0) = \int_{-\infty}^{\infty} e^{-\beta w^2}\, dw.$

Let $\quad z = \sqrt{\beta}\, w \quad \therefore\ dz = \sqrt{\beta}\, dw$

$$\Rightarrow\ g(0) = \frac{1}{\sqrt{\beta}} \int_{-\infty}^{\infty} e^{-z^2}\, dz.$$

By Calculus, $\quad I = \int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi}$

Why? $\quad I^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\,dy$$

~~Using~~ Consider x-y plane and use the polar coordinates.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} re^{-r^2} dr$$

$$\Rightarrow I^2 = 2\pi \cdot \frac{1}{2} = \pi.$$

Another way (Simpler) is to use the method of complex variables.

Fourier transforms and Heat equations

Recall, the general solution of the heat equation on real line is solved by

$$u(x,t) = \int_{-\infty}^{\infty} C(\omega) e^{-i\omega x} e^{-k\omega^2 t} d\omega.$$

w/ the initial condition

$$f(x) = \int_{-\infty}^{\infty} C(\omega) e^{-i\omega x} dx$$

Here $C(\omega)$ is the Fourier transform of the initial temperature distribution $f(x)$.

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$$

Plug in the coefficient formula.

$$u(x,t) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(y) e^{iwy}\, dy\right] e^{-iwx} e^{-kw^2t}\, dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y) \left[\int_{-\infty}^{\infty} e^{-iw(x-y)} e^{-kw^2t}\, dw\right] dy$$

Hence, we just need to evaluate the inverse Fourier transform of e^{-kw^2t}

Let $g(x) = \int_{-\infty}^{\infty} e^{-kw^2t} e^{-iwx}\, dx$

then $u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, \bar{F}(g(x-\bar{x}))\, d\bar{x})$

By the inverse Fourier transform of Gaussian, we get

$$u(x,t) = \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt}\, dy$$

Let $G(x,t,y,0) = \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt}$
: influence function

$\lim_{t\to 0^+} G(x,t,y,0) = \delta(x-y)$: delta function.

Fundamental solutions

Consider ~~the~~ an initial condition centered at $x=0$.
$u(x,0) = f(x) = \delta(x)$.

Properties of $\delta(x)$: $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\, dx = f(x_0).$$

$$u(x,t) = \int_{-\infty}^{\infty} \delta(y) \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt} dy = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}$$

$\Rightarrow$ fundamental solution, also the Green's function for the heat equations.

Ex.

Consider $u(x,0) = f(x) = \begin{cases} 0 & x<0 \\ 100 & x>0. \end{cases}$

$$u(x,t) = \frac{100}{\sqrt{4\pi kt}} \int_0^{\infty} e^{-(x-y)^2/4kt} dy$$

Let $z = (y-x)/\sqrt{4kt}$ $\quad (dz) = dy/\sqrt{4kt}$

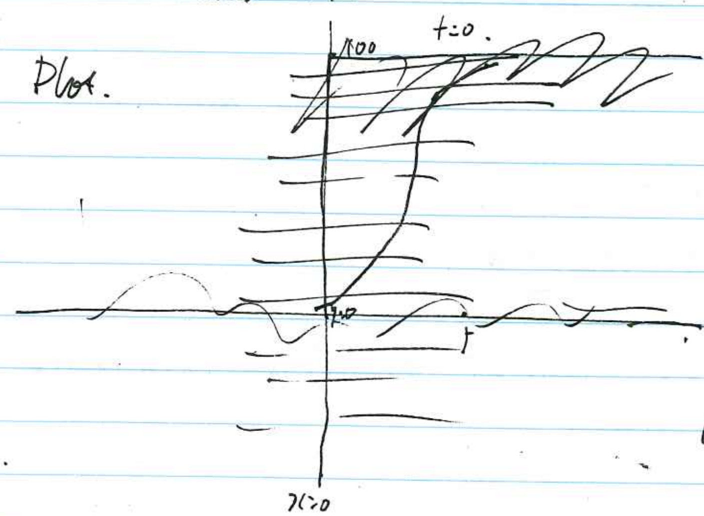
$$u(x,t) = \frac{100}{\sqrt{\pi}} \int_{-x/\sqrt{4kt}}^{\infty} e^{-z^2} dz$$

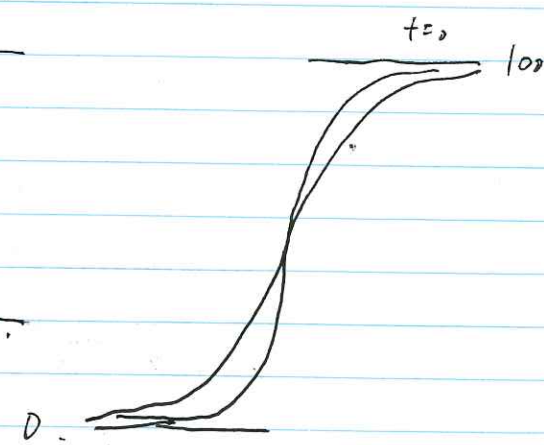
note that $\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$
and e^{-x^2} is even, $\int_0^{\infty} e^{-z^2} dz = \sqrt{\pi}/2$

$$\therefore \quad u(x,t) = 50 + \frac{100}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-z^2} dz$$

Note: the temperature is constant whenever $x/\sqrt{4kt}$ is constant.

Plot.





Fourier transforms of derivatives.

$$\begin{cases} \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} & -\infty < x < \infty \\ u(x,0) = f(x). \end{cases}$$

Without separation of variables.

$$\mathcal{F}\left(\frac{\partial u}{\partial t}\right) = k\, \mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right).$$

where $\mathcal{F}(u)$ is the spatial Fourier transform.

$$\mathcal{F}\left(\frac{\partial u}{\partial t}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{i\omega x}\, dx = \frac{\partial}{\partial t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t) e^{i\omega x}\, dx\right]$$

$$\mathcal{F}\left(\frac{\partial u}{\partial x}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{i\omega x}\, dx = \frac{u e^{i\omega x}}{2\pi}\Big|_{-\infty}^{\infty} - \frac{i\omega}{2\pi}\int_{-\infty}^{\infty} u(x,t) \cdot e^{i\omega x} dx$$

if $u \to 0$ as $x \to \pm\infty$, then the first term is 0
and $\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = -i\omega\, \mathcal{F}[u]$

Similarly, $\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = (-i\omega)^2\, \mathcal{F}[u]$

write $\mathcal{F}[u] = \hat{u}(\omega,t)$, then the heat equation becomes

$$\frac{\partial \hat{u}}{\partial t} = k(-i\omega)^2\, \hat{u} = -k\omega^2\, \hat{u}$$

$$\Rightarrow \quad \hat{u}(\omega,t) = C e^{-k\omega^2 t}$$

by initial condition we can easily get
$\hat{u} = C(\omega) e^{-k\omega^2 t}$ since $\hat{u}(\omega,0) = \hat{f}(\omega) = C(\omega)$.

Remark: The Fourier transform turns ~~an~~ a PDE into an ODE.