Math 241-910 Calculus IV
Summer 2017
Quiz 3
6/13/2017
Time Limit: 60 Minutes

Name: ______________________

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This exam contains 6 pages (including this cover page) and 4 questions.

Total of points is 60.

Grade Table (for teacher use only)

Question	Points	Score
1	15	
2	15	
3	15	
4	15	
Total:	60	

1. (15 points) Consider the equation

$$\phi''(x) - 2\phi'(x) + \lambda(1+x)\phi(x) = 0, \quad 0 \le x \le 2 \tag{1}$$

with boundary conditions

$$\phi'(0) = \phi(2) = 2 \tag{2}$$

1. Is this a regular Strum Liouville problem? What are the $p(x), q(x), \sigma(x)$ respectively?
2. Show that $\lambda \ge 0$.
3. Is $\lambda = 0$ an eigenvalue?
4. If

$$a_n = \int_0^2 (3x^2+1)(1+x)e^{-2x}\phi_n(x)dx \tag{3}$$

calculate

$$\sum_{i=1}^{\infty} a_n \cdot \phi_n(1). \tag{4}$$

2. (15 points) Consider a Sturm-Liouville eigenvalue problem

$$(p\phi')' + q\phi + \lambda\sigma\phi = 0, \quad x \in [1, 2] \tag{5}$$

with boundary conditions

$$\phi(1) = 0 = \phi'(2). \tag{6}$$

Prove the orthogonality of eigenfunctions of this problem. (HINT: *use Green's formula*).

3. (15 points) Let $\omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Let $u(x, y, z, t)$ be a solution to the wave equation

$$u_{tt} = \Delta u - u \quad \text{in } \Omega \tag{7}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{8}$$

where $\Delta u = u_{xx} + u_{yy} + u_{zz}$. Define the energy $E(t)$ of this solution by

$$E(t) = \frac{1}{2} \iiint_{\Omega} u_t^2 + |\nabla u|^2 + u^2 dV. \tag{9}$$

1. Show that $E(t)$ is a constant.
2. Use a) to prove that a solution to the above wave equation with given initial condition ($u(x, y, z, 0)$ and $u_t(x, y, z, 0)$) is unique. Hint: *Suppose that u and v are two such solutions, and study the energy of $w = u - v$.*

4. (15 points) The vertical displacement $u(r\theta, t)$ of a vibrating circular membrane of radius $r_0 = 1$,fixed at the boundary, satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \quad r \in [0,1], \theta \in [-\pi, \pi], t \geq 0 \tag{10}$$

$$u(1, \theta, t) = 0) \tag{11}$$

Let $J_0(z)$ be the 0-th Bessel function of the first kind, and denote its zeroes by $z_{01} < z_{02} < z_{03} < \cdots < z_{0m} < \cdots$. Find a formula for the solution $u(r, \theta, t)$ subject to the initial conditions

$$u(r, \theta, 0) = 0, \quad u_t(r, \theta, 0) = \sum_{m=1}^{\infty} \frac{1}{m^2} J_0(z_{0m} r). \tag{12}$$

You may assume without explanations that separating variables $u(r, \theta, t) = R(r)V(\theta)T(t)$ for the above PDE gives the following ODEs:

$$T'' + \lambda T = 0, \quad V'' + \mu V = 0, \quad r(rR')' + (\lambda r^2 - \mu)R = 0, \quad \lambda \geq 0, \mu \geq 0. \tag{13}$$

Some useful formulas

1. *Rayleigh quotient of regular S-L*:

$$\lambda = \frac{-p\phi\phi'|_a^b + \int_a^b [p\phi'^2 - q\phi^2]dx}{\int_a^b \phi^2 \sigma dx} \tag{14}$$

2. *Rayleigh quotient of Helmholtz equation*:

$$\lambda = \frac{-\oint \phi \nabla\phi \cdot \hat{n} dx + \iint_R |\nabla\phi|^2 dxdy}{\iint_R \phi^2 dx} \tag{15}$$

3. *Green's formula for regular S-L*:

$$\int_a^b \big(uL(v) - vL(u)\big)dx = p(uv' - u'v)\big|_a^b \tag{16}$$

where $L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$.

4. *Bessel functions*:

- **1st kind of order m**: $J_m(z)$. For small z (i.e. $z \to 0$)

$$J_m(z) \sim \begin{cases} 1 & m = 0 \\ \frac{1}{2^m m!} z^m & m > 0 \end{cases} \tag{17}$$

- **2nd kind of order m**: $Y_m(z)$. For small z (i.e. $z \to 0$)

$$J_m(z) \sim \begin{cases} \frac{2}{\pi} \ln z & m = 0 \\ -\frac{2^m (m-1)!}{\pi} z^{-m} & m > 0 \end{cases} \tag{18}$$