**Sturm-Liouville Eigenvalue Problems**

**Motivation**

The heat flow in a nonuniform rod is modeled by the partial differential equation

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q \tag{1}$$

where the thermal coefficients $c, \rho, K_0$ are functions of $x$. If we further assume that the heat source $Q$ is proportional to the temperature $u$, $Q = \alpha(x)u$, then (1) is written

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + \alpha u \tag{2}$$

Assuming that homogeneous boundary conditions are specified, the method of separation of variables may be used to solve (2). If we consider

$$u(x,t) = \Phi(x)G(t) \tag{3}$$

then the variables may be separated

$$\frac{1}{G}\frac{dG}{dt} = \frac{1}{c\rho\Phi}\frac{d}{dx}\left(K_0 \frac{d\Phi}{dx}\right) + \frac{\alpha}{c\rho} = -\lambda \tag{4}$$

such that we obtain the differential equations

$$\frac{dG}{dt} = -\lambda G \tag{5}$$

$$\frac{d}{dx}\left(K_0 \frac{d\Phi}{dx}\right) + \alpha\Phi + \lambda c\rho\Phi = 0 \tag{6}$$

This motivates the study of a general class of differential equations

$$\frac{d}{dx}\left(p \frac{d\Phi}{dx}\right) + q\Phi + \lambda\sigma\Phi = 0 \tag{7}$$

with appropriate boundary conditions. Equation (7) is known as a *Sturm-Liouville differential equation.*

Various types of linear homogeneous boundary conditions may be specified:

- First kind (Dirichlet): $\Phi = 0$
- Second kind (Neumann): $\frac{d\Phi}{dx} = 0$
- Third kind (Robin): $\frac{d\Phi}{dx} = \pm h\Phi$
- Periodicity condition: $\Phi(-L) = \Phi(L)$; $\frac{d\Phi}{dx}(-L) = \frac{d\Phi}{dx}(L)$
- Singularity condition: $|\Phi(0)| < \infty$

A *Sturm-Liouville eigenvalue problem* consists of the Sturm-Liouville differential equation

$$\frac{d}{dx}\left(p \frac{d\Phi}{dx}\right) + q\Phi + \lambda\sigma\Phi = 0, \;\; a < x < b \tag{8}$$

subject to the boundary conditions of the type

$$\beta_1 \Phi(a) + \beta_2 \frac{d\Phi}{dx}(a) = 0 \tag{9}$$

$$\beta_3 \Phi(b) + \beta_4 \frac{d\Phi}{dx}(b) = 0 \tag{10}$$

where $\beta_1, \beta_2, \beta_3, \beta_4$ are real constants. Notice that the periodicity condition is not of the form (9-10). The Sturm-Liouville eigenvalue problem (8), (9-10) is called *regular* if the coefficients $p, q, \sigma$ are real and continuous in $[a, b]$ and $p(x) > 0, \sigma(x) > 0$ for all $x \in [a, b]$.

For any regular Sturm-Liouville problem, the following theorems are valid:

1. All the eigenvalue are real
2. There exists an infinite number of eigenvalues

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots$$

   a. There is a smallest eigenvalue, $\lambda_1$.
   b. There is not a largest eigenvalue and $\lambda_n \to \infty$ as $n \to \infty$

3. Corresponding to each eigenvalue $\lambda_n$, there is an eigenfunction $\Phi_n(x)$ which is unique up to an arbitrary multiplicative constant. In addition, $\Phi_n(x)$ has exactly $n-1$ zeros in the interval $(a, b)$.
4. Any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions

$$f(x) \sim \sum_{n=1}^{\infty} a_n \Phi_n(x)$$

   that converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ if the coefficients are $a_n$ are properly selected.

5. Eigenfunctions corresponding to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$

$$\int_a^b \Phi_n(x)\Phi_m(x)\sigma(x)dx = 0 \quad \text{if} \quad \lambda_n \neq \lambda_m \tag{11}$$

6. Any eigenvalue can be related to its eigenfunction by the *Rayleigh quotient*

$$\lambda = \frac{-p\Phi \left.\frac{d\Phi}{dx}\right|_a^b + \int_a^b \left[p\left(\frac{d\Phi}{dx}\right)^2 - q\Phi^2\right]dx}{\int_a^b \Phi^2 \sigma dx} \tag{12}$$

**Q:** Prove the Rayleigh quotient relation (12).

**Q:** Prove the orthogonality property (11).

*A simple example*

Consider the Sturm-Liouville problem

$$\frac{d^2\Phi}{dx^2} + \lambda\Phi = 0 \tag{13}$$
$$\Phi(0) = 0 \tag{14}$$
$$\Phi(L) = 0 \tag{15}$$

This is a particular case of the Sturm-Liouville problem (8), (9-10) with the coefficients $p(x) \equiv 1, q(x) \equiv 0, \sigma(x) \equiv 1, \beta_1 = 1, \beta_2 = 0, \beta_3 = 1, \beta_4 = 0$. We know that the real eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \ n = 1, 2, \ldots$$

and the corresponding eigenfunctions are

$$\Phi_n(x) = \sin\frac{n\pi x}{L},\ n = 1, 2, \ldots$$

**Q:** Prove Theorem 1 (all eigenvalues are real) for problem (13-15).

*Series of eigenfunctions*

Theorem 4 shows that any piecewise smooth function may be represented by a generalized Fourier series of the eigenfunctions

$$f(x) \sim \sum_{n=1}^{\infty} a_n \Phi_n(x) \tag{16}$$

**Q:** Using the orthogonality of the eigenfunctions (Theorem 5) show that the generalized Fourier coefficients are

$$a_m = \frac{\int_a^b f(x)\Phi_m(x)\sigma(x)dx}{\int_a^b \Phi_m^2(x)\sigma(x)dx}$$

**Q:** For the example (13-15) show that Theorem 6 directly imply that all eigenvalues are positive ($\lambda > 0$).

*Another example: heat flow in a nonuniform rod without sources*

Consider the problem

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) \tag{17}$$

$$u(0,t) \quad = \quad 0 \tag{18}$$

$$\frac{\partial u}{\partial x}(L,t) = \quad = \quad 0 \tag{19}$$

$$u(x,0) = f(x) \tag{20}$$

where the thermal coefficients $c, \rho, K_0$ are functions of $x$.
Using separation of variables

$$u(x,t) = \Phi(x)G(t)$$

we obtain for the time dependent component the differential equation

$$\frac{dG}{dt} = -\lambda G \Rightarrow G(t) = ce^{-\lambda t}$$

and for the space dependent component the Sturm-Liouville eigenvalue problem

$$\frac{d}{dx}\left(K_0\frac{d\Phi}{dx}\right) + \lambda c\rho\Phi = 0 \tag{21}$$

$$\Phi(0) \quad = \quad 0 \tag{22}$$

$$\frac{d\Phi}{dx}(L) \quad = \quad 0 \tag{23}$$

For this problem we know that there is an infinite sequence of eigenvalues $\lambda_n$.

**Q:** Using the Rayleigh quotient, prove that all eigenvalues are positive.

If $\Phi_n$ is the eigenfunction corresponding to $\lambda_n$, then we obtain product solutions of the form

$$u(x,t) = \Phi_n(x) c e^{-\lambda_n t}$$

and using the principle of superposition we write the general solution as

$$u(x,t) = \sum_{n=1}^{\infty} a_n \Phi_n(x) e^{-\lambda_n t}$$

To satisfy the initial condition (20) we must have

$$f(x) = \sum_{n=1}^{\infty} a_n \Phi_n(x)$$

and using the orthogonality of the eigenfunctions we obtain the generalized Fourier coefficients

$$a_n = \frac{\int_0^L f(x)\Phi_n(x)c(x)\rho(x)dx}{\int_0^L \Phi_n^2(x)c(x)\rho(x)dx}$$

**Self-adjoint operators**

Consider the *Sturm-Liouville operator*

$$L(\Phi) = \frac{d}{dx}\left(p\frac{d\Phi}{dx}\right) + q\Phi \tag{24}$$

as a linear operator defined on the space of functions $\Phi(x)$ that satisfy the homogeneous boundary conditions

$$\beta_1\Phi(a) + \beta_2\frac{d\Phi}{dx}(a) = 0 \qquad \beta_3\Phi(b) + \beta_4\frac{d\Phi}{dx}(b) = 0 \tag{25}$$

The Sturm-Liouville eigenvalue problem is then written

$$L(\Phi) + \lambda\sigma(x)\Phi = 0 \tag{26}$$

and properties of the eigenvalues and eigenfunctions are obtained from the study of the operator (24).

**Q:** Show that for any smooth functions $u(x)$ and $v(x)$ the following properties hold:

- Lagrange's identity (differential form)

$$uL(v) - vL(u) = \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right] \tag{27}$$

- Green's formula (integral form of Lagrange's identity)

$$\int_a^b [uL(v) - vL(u)]\,dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_a^b \tag{28}$$

DEFINITION: An operator $L$ is called self-adjoint if for any functions $u$ and $v$ in the domain of definition

$$\int_a^b [uL(v) - vL(u)]\,dx = 0 \tag{29}$$

**Q:** Show that the operator (24) is self-adjoint on the space of functions that satisfy boundary conditions (25).

**Orthogonal eigenfunctions**: eigenfunctions corresponding to distinct eigenvalues are orthogonal

$$L(\Phi_n) + \lambda_n \sigma(x)\Phi_n = 0 \quad | \cdot \Phi_m \quad \int_a^b$$

$$L(\Phi_m) + \lambda_m \sigma(x)\Phi_m = 0 \quad | \cdot \Phi_n \quad \int_a^b$$

After subtracting the relations above and using the fact that $L$ is self-adjoint, we get

$$(\lambda_m - \lambda_n) \int_a^b \Phi_n \Phi_m \sigma(x)\, d\, x = 0 \tag{30}$$

and since $\lambda_m \neq \lambda_m$, the eigenfunctions must be orthogonal

$$\int_a^b \Phi_n \Phi_m \sigma(x)\, d\, x = 0 \tag{31}$$

**Real eigenvalues**: the eigenvalues of a self-adjoint operator are real.

Proof by contradiction, assume that

$$L(\Phi) = \lambda \sigma(x)\Phi$$

where $\lambda$ is a complex number. Then the complex conjugate $\overline{\lambda}$ is also an eigenvalue with the corresponding eigenfunction $\overline{\Phi}$

$$L(\overline{\Phi}) = \overline{\lambda}\sigma(x)\overline{\Phi}$$

Using the relation (30) for $\Phi_n = \Phi, \Phi_m = \overline{\Phi}$ we get

$$(\lambda - \overline{\lambda}) \int_a^b \Phi\overline{\Phi}\sigma(x)\, d\, x = 0 \tag{32}$$

Since $\Phi\overline{\Phi} = |\Phi|^2$ and $\sigma(x) > 0$ the equation above implies $\lambda = \overline{\lambda}$ such that *the eigenvalues are real.*

**Unique eigenfunctions**: The eigenfunctions associated to an eigenvalue are unique, up to a multiplicative constant (e.g., the eigenspace associated to each eigenvalue is of dimension one).

Proof: assume that $\Phi_1$ and $\Phi_2$ are eigenfunctions associated to the same eigenvalue $\lambda$:

$$L(\Phi_1) + \lambda\sigma(x)\Phi_1 = 0 \qquad | \cdot \Phi_2$$

$$L(\Phi_2) + \lambda\sigma(x)\Phi_2 = 0 \qquad | \cdot \Phi_1$$

After subtracting the relations above we get

$$\Phi_2 L(\Phi_1) - \Phi_1 L(\Phi_2) = 0 \tag{33}$$

**Q:** Using Lagrange's identity (27) show that (33) (25) imply

$$\frac{d}{d\,x}\left(\Phi_2/\Phi_1\right) = 0$$

such that

$$\Phi_2 = c\Phi_1$$

and the eigenfunctions are linearly dependent.

**Section 5.6: Rayleigh quotient and the minimization principle**

Consider the eigenvalue problem

$$L\phi(x) + \lambda\sigma(x)\phi(x) = 0, \quad a < x < b \tag{34}$$

where $L$ is a Sturm-Liouville operator,

$$L\phi = [p(x)\phi'(x)]' + q(x)\phi(x)$$

defined for functions that satisfy the boundary conditions

$$\beta_1\phi(a) + \beta_2\phi'(a) = 0, \quad \beta_3\phi(b) + \beta_4\phi'(b) = 0 \tag{35}$$

The Rayleigh quotient provides the eigenvalues in terms of the eigenfunctions, as

$$\lambda = \frac{-\int_a^b \phi L\phi \, dx}{\int_a^b \sigma\phi^2 \, dx} \tag{36}$$

For an arbitrary function $u \in C^1([a,b])$ that satisfies the boundary conditions (35) we define the Rayleigh quotient as

$$\mathcal{R}(u) = \frac{-\int_a^b uLu \, dx}{\int_a^b \sigma u^2 \, dx} \tag{37}$$

Next we show that the lowest (first) eigenvalue satisfies the **minimization principle**

$$\lambda_1 = \min_u \mathcal{R}(u) \tag{38}$$

**Proof:** Given a function $u$, we consider the series representation

$$u(x) = \sum_{n=1}^{\infty} a_n\phi_n(x)$$

We know that eigenfunctions corresponding to distinct eigenvalues are orthogonal and we may assume that in addition $\phi_n$ are *orthonormal*

$$\int_a^b \sigma\phi_n\phi_m \, dx = 0 \text{ if } m \neq n, \quad \int_a^b \sigma\phi_n^2 \, dx = 1 \tag{39}$$

The Rayleigh quotient (37) is then written

$$\mathcal{R}(u) = \frac{-\int_a^b \left(\sum_{n=1}^{\infty} a_n\phi_n\right)\left(-\sum_{n=1}^{\infty} a_n\lambda_n\sigma\phi_n\right) dx}{\int_a^b \left(\sum_{n=1}^{\infty} a_n\sigma\phi_n\right)\left(\sum_{n=1}^{\infty} a_n\phi_n\right) dx}$$

and using the orthogonality of the eigenfunctions (39) we may simplify the relation above to

$$\mathcal{R}(u) = \frac{\sum_{n=1}^{\infty} \lambda_n a_n^2}{\sum_{n=1}^{\infty} a_n^2}$$

If the eigenvalues are ordered such that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$ then $\lambda_n a_n^2 \geq \lambda_1 a_n^2$ such that from the equation above we get

$$\mathcal{R}(u) \geq \lambda_1 \frac{\sum_{n=1}^{\infty} a_n^2}{\sum_{n=1}^{\infty} a_n^2} = \lambda_1$$

which proves the minimization principle (38). Notice that equality can be achieved only if $\lambda_n a_n^2 = \lambda_1 a_n^2$ for all $n \geq 2$ which is possible if and only if $a_n = 0, n \geq 2$ such that the Rayleigh quotient (37) is minimized only when $u = a_1\phi_1$.

**Q:** The minimization principle can be generalized to higher rank eigenvalues. If we define the spaces of functions

$$\mathcal{V} = \{u \in C^1([a,b]), \quad \beta_1 u(a) + \beta_2 u'(a) = 0, \quad \beta_3 u(b) + \beta_4 u'(b) = 0\}$$

$$\mathcal{V}_k = \{u \in \mathcal{V}, \int_a^b \sigma u \phi_i \, dx = 0, \quad i = 1, 2, \ldots k\}$$

prove that the following minimization property holds:

$$\lambda_{k+1} = \min_{u \in \mathcal{V}_k} \mathcal{R}(u), \quad k \geq 1 \tag{40}$$

**Q:** Prove that if $u \in C^2([a,b])$ solves the minimization problem

$$\min_{u \in \mathcal{V}} \left[ -\frac{1}{2} \int_a^b uLu \, dx + \int_a^b uf \, dx \right]$$

then $u$ solves the nonhomogeneous problem

$$Lu = f$$

**Q:** Consider the Poisson problem in a domain $\Omega \subset R^n$ with boundary $S$

$$\Delta u = f, \quad x \in \Omega \tag{41}$$

$$u = 0, \quad x \in S \tag{42}$$

What minimization problem does $u$ solve? Give a physical interpretation.