The Laplace equation on a solid cylinder

The next problem we'll consider is the solution of Laplace's equation $\nabla^2 u = 0$ on a solid cylinder. We'll do this in cylindrical coordinates, which of course are the just polar coordinates (r, θ) replacing (x, y) together with z. We'll let our cylinder have height H and radius a, so the z coordinate will go from 0 to H, and the r coordinate will go from 0 to a.

The entire problem can be written

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

for $0 \le r \le a$, $0 \le \theta \le 2\pi$ and $0 \le z \le H$ together with boundary conditions

$$\begin{aligned} u(r, \theta, H) &= f(r, \theta) \quad \text{on the top} \\ u(r, \theta, 0) &= g(r, \theta) \quad \text{on the bottom} \\ u(a, \theta, z) &= h(\theta, z) \quad \text{on the side} \end{aligned}$$

As we did for the Laplace equation on a rectangle, we'll break this into three separate problems, where we set two of the boundary conditions equal to zero in each problem, and then add the three partial solutions together at the end.

As usual, we'll start by separating the variables. We look for product solutions in the form

$$u(r, \theta, z) = R(r)\Theta(\theta)Z(z).$$

Putting this into the differential equation and dividing by $R\Theta Z$ gives

$$\frac{(rR')'}{rR} + \frac{\Theta''}{r^2\Theta} + \frac{Z''}{Z} = 0$$

We can separate the Z part right off:

$$\frac{(rR')'}{rR} + \frac{\Theta''}{r^2\Theta} = -\frac{Z''}{Z} = -\lambda$$

where λ is a constant because the left side is a function of r and θ alone and the right side is a function of z alone (and we choose the minus sign on λ to agree with the notation in the textbook), so

$$Z'' - \lambda Z = 0.$$

Next, we multiply the part involving R and Θ by r^2 to get

$$\frac{r^2 R'' + rR' + \lambda r^2}{R} = -\frac{\Theta''}{\Theta} = \mu$$

where μ is another constant.

The Θ equation is thus

$$\Theta'' + \mu\Theta = 0,$$

and since Θ must be periodic with period 2π, we must have $\mu = n^2$ for $n = 0, 1, 2, \ldots$ and Θ is a linear combination of $\cos n\theta$ and $\sin n\theta$.

Finally, the R equation is

$$r^2 R'' + rR' + (\lambda r^2 - n^2)R = 0$$

which looks a little like Bessel's equation, but there will be a twist. Now we're ready to do our three-part problem

Part I. Non-zero boundary values only on the top (i.e., $g = 0$ and $h = 0$)

Let $u_1(r, \theta, z)$ be the solution of the sub-problem for which the functions on the bottom and the side are zero, i.e., $u_1(r, \theta, 0) = 0$ and $u_1(a, \theta, z) = 0$, but $u_1(r, \theta, H) = f(r, \theta)$, then we have the boundary conditions $Z(0) = 0$ and $R(a) = 0$. We'll solve for R first — no twist here, we have that

$$R(r) = J_n(\sqrt{\lambda}\, r)$$

and we need this to be zero when $r = a$. So $\sqrt{\lambda}a = z_{nm}$ and so

$$\lambda = \left(\frac{z_{nm}}{a}\right)^2 \quad \text{and} \quad R(r) = J_n\left(\frac{z_{nm} r}{a}\right).$$

Now that we know λ, we put it into the Z equation to get

$$Z'' - \left(\frac{z_{nm}}{a}\right)^2 Z = 0.$$

This has exponential solutions, but its easier to write them as hyperbolic functions: since $Z(0) = 0$ we have

$$Z = \sinh\left(\frac{z_{nm} z}{a}\right)$$

and we have for the solution of the Part I problem:

$$u_1(r, \theta, z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} J_n\left(\frac{z_{nm} r}{a}\right) \sinh\left(\frac{z_{nm} z}{a}\right) [a_{nm} \cos n\theta + b_{nm} \sin n\theta].$$

To calculate the coefficients , note that we need

$$f(r, \theta) = u_1(r, \theta, H) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} J_n\left(\frac{z_{nm} r}{a}\right) \sinh\left(\frac{z_{nm} H}{a}\right) [a_{nm} \cos n\theta + b_{nm} \sin n\theta]$$

If we view θ as the variable and r as constant for the moment, this becomes an ordinary Fourier series for $f(r,\theta)$, so we have

$$\sum_{m=1}^{\infty} a_{0m} J_0\left(\frac{z_{0m} r}{a}\right) \sinh\left(\frac{z_{0m} H}{a}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(r,\theta)\, d\theta \quad \text{for } n = 0,$$
$$\sum_{m=1}^{\infty} a_{nm} J_n\left(\frac{z_{nm} r}{a}\right) \sinh\left(\frac{z_{nm} H}{a}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(r,\theta)\cos m\theta\, d\theta \quad \text{for } n \geq 1,$$
$$\sum_{m=1}^{\infty} b_{nm} J_n\left(\frac{z_{nm} r}{a}\right) \sinh\left(\frac{z_{nm} H}{a}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(r,\theta)\sin m\theta\, d\theta \quad \text{for } n \geq 1.$$

But the left sides of these are Fourier-Bessel series, so using the results of the notes on the wave equation on the disk we finally obtain the coefficients:

$$a_{0m} = \frac{\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\int_0^a r f(r,\theta) J_0\left(\frac{z_{0m} r}{a}\right) dr\, d\theta}{\sinh\left(\dfrac{z_{0m} H}{a}\right)\displaystyle\int_0^a r J_0\left(\frac{z_{0m} r}{a}\right)^2 dr} \quad \text{for } n = 0, m \geq 1$$

$$a_{nm} = \frac{\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\int_0^a r f(r,\theta) J_n\left(\frac{z_{nm} r}{a}\right) \cos n\theta\, dr\, d\theta}{\sinh\left(\dfrac{z_{nm} H}{a}\right)\displaystyle\int_0^a r J_n\left(\frac{z_{nm} r}{a}\right)^2 dr} \quad \text{for } n \geq 1, m \geq 1$$

$$b_{nm} = \frac{\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\int_0^a r f(r,\theta) J_n\left(\frac{z_{nm} r}{a}\right) \sin n\theta\, dr\, d\theta}{\sinh\left(\dfrac{z_{nm} H}{a}\right)\displaystyle\int_0^a r J_n\left(\frac{z_{nm} r}{a}\right)^2 dr} \quad \text{for } n \geq 1, m \geq 1$$

and the integral in the denominators is given by

$$\int_0^a r J_n\left(\frac{z_{nm}}{a} r\right)^2 dr = \frac{a^2}{2} J_{n+1}(z_{nm})^2.$$

Part II. Non-zero boundary values only on the bottom (i.e., $f = 0$ and $h = 0$)

Let $u_2(r,\theta,z)$ be the solution of the sub-problem for which the functions on the top and the side are zero, i.e., $u_2(r,\theta,H) = 0$ and $u_2(a,\theta,z) = 0$, but $u_2(r,\theta,0) = g(r,\theta)$, then we have the boundary conditions $Z(H) = 0$ and $R(a) = 0$. Once again, we'll solve for R first — and get the same answer. We have that

$$R(r) = J_n(\sqrt{\lambda}\, r)$$

and we need this to be zero when $r = a$. So $\sqrt{\lambda}a = z_{nm}$ and so

$$\lambda = \left(\frac{z_{nm}}{a}\right)^2 \quad \text{and} \quad R(r) = J_n\left(\frac{z_{nm}r}{a}\right).$$

As in part I, now that we know λ, we put it into the Z equation to get

$$Z'' - \left(\frac{z_{nm}}{a}\right)^2 Z = 0.$$

This has exponential solutions, and its again easier to write them as hyperbolic functions: since $Z(H) = 0$ we have

$$Z = \sinh\left(\frac{z_{nm}(H - z)}{a}\right)$$

and we have for the solution of the Part II problem:

$$u_2(r, \theta, z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{z_{nm}r}{a}\right)\sinh\left(\frac{z_{nm}(H - z)}{a}\right)[c_{nm}\cos n\theta + d_{nm}\sin n\theta].$$

To calculate the coefficients , note that we need

$$g(r, \theta) = u_2(r, \theta, 0) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{z_{nm}r}{a}\right)\sinh\left(\frac{z_{nm}H}{a}\right)[c_{nm}\cos n\theta + d_{nm}\sin n\theta]$$

This is the same series as in Part I, so we proceed the same way. If we view θ as the variable and r as constant for the moment, this becomes an ordinary Fourier series for $g(r, \theta)$, so we have

$$\begin{aligned}
\sum_{m=1}^{\infty} c_{0m}J_0\left(\frac{z_{0m}r}{a}\right)\sinh\left(\frac{z_{0m}H}{a}\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(r, \theta)\,d\theta \quad \text{for } n = 0,\\
\sum_{m=1}^{\infty} c_{nm}J_n\left(\frac{z_{nm}r}{a}\right)\sinh\left(\frac{z_{nm}H}{a}\right) &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(r, \theta)\cos m\theta\,d\theta \quad \text{for } n \geq 1,\\
\sum_{m=1}^{\infty} d_{nm}J_n\left(\frac{z_{nm}r}{a}\right)\sinh\left(\frac{z_{nm}H}{a}\right) &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(r, \theta)\sin m\theta\,d\theta \quad \text{for } n \geq 1.
\end{aligned}$$

The left sides of these are Fourier-Bessel series, so using the results of the notes on the wave equation on the disk we finally obtain the coefficients:

$$c_{0m} = \frac{\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\int_0^a rg(r,\theta)J_0\left(\frac{z_{0m}r}{a}\right)\,dr\,d\theta}{\sinh\left(\dfrac{z_{0m}H}{a}\right)\displaystyle\int_0^a rJ_0\left(\frac{z_{0m}r}{a}\right)^2\,dr} \quad \text{for } n=0, m\geq 1$$

$$c_{nm} = \frac{\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\int_0^a rg(r,\theta)J_n\left(\frac{z_{nm}r}{a}\right)\cos n\theta\,dr\,d\theta}{\sinh\left(\dfrac{z_{nm}H}{a}\right)\displaystyle\int_0^a rJ_n\left(\frac{z_{nm}r}{a}\right)^2\,dr} \quad \text{for } n\geq 1, m\geq 1$$

$$d_{nm} = \frac{\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\int_0^a rg(r,\theta)J_n\left(\frac{z_{nm}r}{a}\right)\sin n\theta\,dr\,d\theta}{\sinh\left(\dfrac{z_{nm}H}{a}\right)\displaystyle\int_0^a rJ_n\left(\frac{z_{nm}r}{a}\right)^2\,dr} \quad \text{for } n\geq 1, m\geq 1$$

and the integral in the denominators is again given by

$$\int_0^a rJ_n\left(\frac{z_{nm}}{a}r\right)^2\,dr = \frac{a^2}{2}J_{n+1}(z_{nm})^2.$$

Part III. Non-zero boundary values only on the side (i.e., $f=0$ and $g=0$)

Now we come to the new wrinkle. Let $u_3(r,\theta,z)$ be the solution of the sub-problem for which the functions on the top and the bottom are zero, i.e., $u_3(r,\theta,H)=0$ and $u_3(r,\theta,H)=0$, but $u_3(a,\theta,z)=h(\theta,z)$, then we have the boundary conditions $Z(0)=0$ and $Z(H)=0$.

Since the boundary conditions only involve Z, this time we'll solve for Z first. The Z equation started out as

$$Z''-\lambda Z=0$$

and the boundary conditions on Z imply that

$$Z=\sin\left(\frac{m\pi z}{H}\right) \quad \text{and} \quad \lambda=-\left(\frac{m\pi}{H}\right)^2$$

for $m=1,2,3,\ldots$. These values of λ make the R equation, which was

$$r^2R''+rR'+(\lambda r^2-n^2)R=0$$

into

$$r^2R''+rR'+\left(-\frac{m^2\pi^2}{H^2}r^2-n^2\right)R=0$$

This looks like Bessel's equation, except the sign is wrong on the r^2R term. But when we had the R equation with $\sqrt{\lambda}$ in it when we were studying the wave equation on the disk, we learned that if we set $x = \sqrt{\lambda}\, r$, then we obtained Bessel's equation for $R(x)$. So, undaunted by the fact that λ is negative in this case, we set

$$x = \sqrt{\lambda} r = \sqrt{-\left(\frac{m\pi}{H}\right)^2}\, r = i\frac{m\pi}{H}\, r$$

and the R equation becomes

$$x^2 \frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - n^2)R = 0$$

for which the only solutions bounded at $r = x = 0$ are constant multiples of

$$J_n(x) = J_n\left(i\frac{m\pi}{H}\, r\right).$$

Now, having a complex answer might not seem to useful, but by way of digging a little deeper here, we recall the series for $J_n(x)$:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

If we ignore the $m\pi/H$ part and just substitute ir for x, we get

$$\begin{aligned}
J_n(ir) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{ir}{2}\right)^{n+2k} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{ir}{2}\right)^{n+2k} \\
&= i^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} i^{2k}\left(\frac{r}{2}\right)^{n+2k} \\
&= i^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} (-1)^k\left(\frac{r}{2}\right)^{n+2k} \\
&= i^n \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!}\left(\frac{r}{2}\right)^{n+2k}
\end{aligned}$$

So we have that

$$I_n(r) = \frac{1}{i^n} J_n(ir) = \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!}\left(\frac{r}{2}\right)^{n+2k}$$

is a *real* function, called the *modified Bessel function* of the first kind of order n. Since the power series of $I_n(r)$ has only positive coefficients, we have that I_n is a positive, increasing, concave-up function (rather like the exponential function or $\cosh(x)$ for positive values of r). Indeed, the relationship between J_n and I_n is analogous to the relationship between the trigonometric functions and the hyperbolic functions.

You can check, either by plugging the series for I_n into the differential equation, or else by going through the whole Frobenius power series method starting from the differential equation, that I_n indeed satisfies the differential equation. So we conclude that

$$R(r) = I_n\left(\frac{m\pi}{H}r\right).$$

So we conclude that the solution of the Part III problem is

$$u_3(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} I_n\left(\frac{m\pi r}{H}\right)\sin\left(\frac{m\pi z}{H}\right)[e_{nm}\cos n\theta + f_{nm}\sin n\theta].$$

After all the complication with I_n, it might seem surprising that the coefficients are relatively easy to specify, since, setting $r = a$ to get the last boundary condition gives

$$u_3(a,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} I_n\left(\frac{m\pi a}{H}\right)\sin\left(\frac{m\pi z}{H}\right)[e_{nm}\cos n\theta + f_{nm}\sin n\theta],$$

and this is an ordinary (double) Fourier series — a sine series in z and a full Fourier series in θ. So the coefficients are

$$e_{0m} = \frac{1}{\pi H I_0\left(\frac{m\pi a}{H}\right)}\int_{-\pi}^{\pi}\int_0^H h(\theta,z)\sin\left(\frac{m\pi z}{H}\right)\,dz\,d\theta \quad \text{for } n=0, m\geq 1$$

$$e_{nm} = \frac{2}{\pi H I_n\left(\frac{m\pi a}{H}\right)}\int_{-\pi}^{\pi}\int_0^H h(\theta,z)\sin\left(\frac{m\pi z}{H}\right)\cos n\theta\,dz\,d\theta \quad \text{for } n\geq 1, m\geq 1$$

$$f_{nm} = \frac{2}{\pi H I_n\left(\frac{m\pi a}{H}\right)}\int_{-\pi}^{\pi}\int_0^H h(\theta,z)\sin\left(\frac{m\pi z}{H}\right)\sin n\theta\,dz\,d\theta \quad \text{for } n\geq 1, m\geq 1$$

Putting it all together

Now we have all three pieces of the solution, so we bring them all together to conclude:

The solution of the problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

for $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq H$ together with boundary conditions

$$\begin{aligned} u(r,\theta,H) &= f(r,\theta) \quad \text{on the top} \\ u(r,\theta,0) &= g(r,\theta) \quad \text{on the bottom} \\ u(a,\theta,z) &= h(\theta,z) \quad \text{on the side} \end{aligned}$$

is

$$u(r,\theta,z) = u_1(r,\theta,z) + u_2(r,\theta,z) + u_3(r,\theta,z),$$

where (with z_{nm} being the mth positive zero of the Bessel function $J_n(x)$)

$$u_1(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{z_{nm}r}{a}\right)\sinh\left(\frac{z_{nm}z}{a}\right)[a_{nm}\cos n\theta + b_{nm}\sin n\theta],$$

$$u_2(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{z_{nm}r}{a}\right)\sinh\left(\frac{z_{nm}(H-z)}{a}\right)[c_{nm}\cos n\theta + d_{nm}\sin n\theta],$$

and

$$u_3(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} I_n\left(\frac{m\pi r}{H}\right)\sin\left(\frac{m\pi z}{H}\right)[e_{nm}\cos n\theta + f_{nm}\sin n\theta].$$

The coefficients a_{nm}, b_{nm}, c_{nm}, d_{nm}, e_{nm} and f_{nm} are given by

$$a_{0m} = \frac{\displaystyle\int_{-\pi}^{\pi}\int_0^a rf(r,\theta)J_0\left(\frac{z_{0m}r}{a}\right)\,dr\,d\theta}{\pi a^2 J_1(z_{0m})^2\sinh\left(\dfrac{z_{0m}H}{a}\right)} \quad \text{for } n=0, m\geq 1$$

$$a_{nm} = \frac{2\displaystyle\int_{-\pi}^{\pi}\int_0^a rf(r,\theta)J_n\left(\frac{z_{nm}r}{a}\right)\cos n\theta\,dr\,d\theta}{\pi a^2 J_{n+1}(z_{nm})^2\sinh\left(\dfrac{z_{nm}H}{a}\right)} \quad \text{for } n\geq 0, m\geq 1$$

$$b_{nm} = \frac{2\displaystyle\int_{-\pi}^{\pi}\int_0^a rf(r,\theta)J_n\left(\frac{z_{nm}r}{a}\right)\sin n\theta\,dr\,d\theta}{\pi a^2 J_{n+1}(z_{nm})^2\sinh\left(\dfrac{z_{nm}H}{a}\right)} \quad \text{for } n\geq 0, m\geq 1$$

$$c_{0m} = \frac{\displaystyle\int_{-\pi}^{\pi}\int_0^a rg(r,\theta)J_0\left(\frac{z_{0m}r}{a}\right)\,dr\,d\theta}{\pi a^2 J_1(z_{0m})^2 \sinh\left(\dfrac{z_{0m}H}{a}\right)} \quad \text{for } n=0, m\geq 1$$

$$c_{nm} = \frac{\displaystyle 2\int_{-\pi}^{\pi}\int_0^a rg(r,\theta)J_n\left(\frac{z_{nm}r}{a}\right)\cos n\theta\,dr\,d\theta}{\pi a^2 J_{n+1}(z_{nm})^2 \sinh\left(\dfrac{z_{nm}H}{a}\right)} \quad \text{for } n\geq 1, m\geq 1$$

$$d_{nm} = \frac{\displaystyle 2\int_{-\pi}^{\pi}\int_0^a rg(r,\theta)J_n\left(\frac{z_{nm}r}{a}\right)\sin n\theta\,dr\,d\theta}{\pi a^2 J_{n+1}(z_{nm})^2 \sinh\left(\dfrac{z_{nm}H}{a}\right)} \quad \text{for } n\geq 1, m\geq 1$$

$$e_{0m} = \frac{\displaystyle\int_{-\pi}^{\pi}\int_0^H h(\theta,z)\sin\left(\frac{m\pi z}{H}\right)\,dz\,d\theta}{\pi H I_0\left(\dfrac{m\pi a}{H}\right)} \quad \text{for } n=0, m\geq 1$$

$$e_{nm} = \frac{\displaystyle 2\int_{-\pi}^{\pi}\int_0^H h(\theta,z)\sin\left(\frac{m\pi z}{H}\right)\cos n\theta\,dz\,d\theta}{\pi H I_n\left(\dfrac{m\pi a}{H}\right)} \quad \text{for } n\geq 1, m\geq 1$$

$$f_{nm} = \frac{\displaystyle 2\int_{-\pi}^{\pi}\int_0^H h(\theta,z)\sin\left(\frac{m\pi z}{H}\right)\sin n\theta\,dz\,d\theta}{\pi H I_n\left(\dfrac{m\pi a}{H}\right)} \quad \text{for } n\geq 1, m\geq 1.$$

That's it!