Set. A1. 1. (1) Let $S_n = 1+3+5+\cdots+(2n+1)$

$n=1$ $S_n = 4$ $\Rightarrow$ Guess: $S_n = (n+1)^2$ in general.

$n=2$ $S_n = 9$

$n=3$ $S_n = 16$

Induction: case $n=1$ is clearly true. Suppose the case $n=k-1$ is true,

that is $S_{k-1} = k^2$.

Let's look at S_k. $S_k = S_{k-1} + (2k+1) = k^2+2k+1$

induction hypothesis

$= (k+1)^2$ (by the perfect square formula).

(b). Similar method, first guess the general form of $S_n = 1+2^2+3^2+\cdots+n^2$.

Let see first few terms.

$n=1$, $S_1 = 1$, $n=2$ $S_2 = 5$, $n=3$ $S_3 = 14$

$n=4$ $S_4 = 30$, $n=5$ $S_5 = 55$, $n=6$ $S_6 = 91$

Note $S_1 = 1$

$S_2 = 5 = 1\times5$ — $(2n+1)$ terms.

$S_3 = 14 = 2\times7$

$S_4 = 30 = 2\times3\times5$

$S_5 = 55 = 5\times11$

$S_6 = 91 = 7\times13$

$S_7 = 140 = 2\times2\times5\times7$

$$= \frac{n\,[\,2n^2+5n+3-2n-2\,]}{2}$$

$$= n\left[\,\frac{2n^2+3n-1}{2}\,\right]$$

$$= \frac{n\,(n+1)(2n+1)}{2}$$

Therefore, $S_n = \sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$

#2. Let $\sum_{i=1}^{n} i^3 = S_n$. Clearly when $n=1$, $S_1 = 1 = \dfrac{1\cdot 2^2}{4}$

Suppose the case $n=k$ is true, that is

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$$

Consider when $n=k+1$.

$$S_{k+1} = S_k + (k+1)^3 \underset{\text{induction hypothesis}}{=} \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k^2+4k+4)(k+1)^2}{4} = \frac{(k+2)^2(k+1)^2}{4}$$

~~Hence~~ Which is what we want.

#3. Denote the sum by S_n as usual. $n=1$ case is clearly true. Suppose the case $n=k$ is true, i.e. $\sum_{i=1}^{k} 1/i(i+1) = k/(k+1)$

$$S_{k+1} = S_k + 1/(k+1)(k+2) = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

Another approach: since $1+3+\cdots+(2n-1) = n^2$ by part (a).
We can regard $\sum_{k=1}^{n} k^2$ as

$$S_n \sum_{k=1}^{n} \sum_{i=1}^{k} (2k-1)$$

$$1 = 1$$
$$4 = 1+3$$
$$9 = 1+3+5$$
$$\cdot \quad 1+3+5+7$$
$$\vdots \quad \vdots$$
$$n^2 = 1+3+\cdots\cdots\cdots+(2n-1)$$

Hence, we get

$$S_n = 1\times n + 3\times(n-1) + \cdots + (2n-1)$$
$$= \sum_{k=1}^{n} (2k-1)(n+1-k);$$
$$= \sum_{k=1}^{n} \Big((2k-1)(n+1) - k(2k-1) \Big)$$
$$= \sum_{k=1}^{n} \Big((n+1)(2k-1) + k \Big) - 2 \underbrace{\sum_{k=1}^{n} k^2}_{S_n}$$

Hence
$$3S_n = \sum_{k=1}^{n} \big(2k(n+1) + k\big) - (n+1)n$$
$$= 2n\sum_{k=1}^{n} k + 2\sum_{k=1}^{n} k + \sum_{n=1}^{n} k - (n+1)n$$
$$= (2n+3)\sum_{k=1}^{n} k - (n+1)n$$
$$= (2n+3)\frac{n(n+1)}{2} - (n+1)n$$
$$= \frac{n(2n+3)(n+1) - 2n(n+1)}{2}$$
$$= \frac{n[(2n+3)(n+1) - 2(n+1)]}{2}$$

$$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

as desired.

Remark: There is an easier way (direct approach).
Note that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, since

$$\frac{1}{k} - \frac{1}{k+1} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{1}{k(k+1)}.$$

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n\cdot(n+1)}$$

$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

$= \frac{n}{n+1}$ since all middle terms are cancelled.

#4. Proof goes similar as in proposition A.1.4.
We induct on $n = |T|$. Clearly when $n = 0$ it is true. Suppose the result hold for $n = k$. Now suppose $\varphi : S \to T$ is surjective with $|T| = k+1$. Pick k elements of T and called the set formed by these elements T'. Now $|T'| = k$. Consider the preimage $\varphi^{-1}(T') \subset S$. Then

$\varphi|_{\varphi^{-1}(T')}$ is a surjective map to T',
which implies $|\varphi^{-1}(T')| \geq |T'| = k$ by induction hypothesis. Denote the only element $x \in T$

such that $x \notin T'$. We know $\varphi^{-1}(x)$ cannot have intersection with $\varphi^{-1}(T')$. Also, we know $|\varphi^{-1}(x)| \geq |\{x\}| = 1$ by the case $k=1$.

Hence $|S| \geq |\varphi^{-1}(T') \cup \varphi^{-1}(x)| \geq n+1 = |T|$.

If $|S| = |T|$, then all inequalities above become equalities. By similar induction we can get $|S| = |T| = n$ for any n with $\varphi: S \to T$ surjective, φ must be surjective.

#5. Prove by ~~contradiction~~ contrapositive. Suppose n is not a prime, we want to show $2^n - 1$ is not a prime.

Write n as a product $n = ab$ such that $1 < a, b < n$.

So $2^n - 1 = 2^{ab} - 1 = (2^a)^b - 1$.

Recall the difference of powers formula.

$$x^{p+1} - y^{p+1} = (x-y)(x^p + x^{p-1}y + \cdots + xy^{p-1} + y^p)$$

So
$$
\begin{aligned}
(2^a)^b - 1 &= (2^a)^b - 1^b \\
&= (2^a - 1)((2^a)^{b-1} + (2^a)^{b-2} + \cdots + 1)
\end{aligned}
$$

Since $a > 1$, $(2^a - 1) > 1$, which implies $(2^a)^b - 1$ is a comoposite.

#6. Prove by induction

$a_0 = 2^1 + 1 = 3$ $\quad a_1 = 2^2 + 1 = 5$ $\quad$ So $a_1 = a_0 + 2$

the base case is true. Suppose the case $n = k$ is true

i.e. $a_k = a_0 \cdots a_{k-1} + 2$,

Consider $n = k+1$

$$a_{k+1} = 2^{2^{(k+1)}} + 1 = 2^{2^k \cdot 2} + 1 = (2^{2^k})^2 + 1$$

$$\xleftrightarrow{\text{induction hypothesis}} (a_k - 1)^2 + 1 = (a_0 \cdots a_{k-1} + 1)(a_k - 1) + 1$$

$$= a_0 \cdots a_{k-1} a_k + \overbrace{a_k - a_0 \cdots a_{k-1}}^{2}$$

$$= a_0 \cdots a_{k-1} a_k + 2 \quad \text{as desired.}$$

#7. Induct on the degree of polynomials.

Sec. A2.

#8. ~~By~~ Prove by contradiction.

#9. (a) Let $x \in \mathbb{N}$, let S be the set consists of all $y \in \mathbb{N}$ s.t. $x+y = y+x$. Clearly $0 \in S$. Now if $y \in S$,

$$\begin{aligned} x + y' &= (x+y)' \quad \text{(by A.2.2)} \\ &= (y+x)' \\ &= y + x' \end{aligned}$$

It suffices to show $y + x' = \cancel{x+y'}\ y' + x$

Let $S' = \{ y \in \mathbb{N} : \cancel{x+y' = x'+y}\ y + x' = y' + x \}$.

Clearly $0 \in S'$, assume $y \in S'$,

$$\begin{aligned} y' + x' &= (y'+x)' \\ &= (y+x')' \\ &= y + x'' \quad \Rightarrow S = \mathbb{N}. \end{aligned}$$

Hence we are done.

(b) Similar to the proof of associativity law for addition.

We want to show $abn = a(bn)\ \forall n \in \mathbb{N}$.

$n=1$ case is easy. Suppose $n=k$ works,

$$\begin{aligned} abk' = (ab)k' &= abk + ab \\ &= a(bk) + ab \\ &= a(bk+b) \\ &= a(bk') \end{aligned}$$

(c) $n=1$ $\quad (a+b)\cdot 1 = a+b = a\cdot 1 + b \cdot 1$