Week 4.

Small groups : we use Larange's theorem to study the structures of some groups of small orders.

Ex.1. D_{10}: any subgroups of D_{10} must have order 1, 2, 5, 10.

1: $\{e\}$.

2: 5 reflections

5: $\langle r\rangle$.

Exercise: How about D_8?

1: $\{e\}$.

2. 4 reflections and r^2.

4. $\langle r\rangle$, there are two other non-cyclic.

8. D_8.

Prop: Any group of order 4 is either isomorphic to C_4 or ~~$C_4\times C_4$~~ $C_2\times C_2$.

Pf: Later.

Similarly, any group of order 6 is either isomorphic to D_6 or C_6.

Left / right cosets.

Let $G = D_6 = \langle r, s \mid r^3 = e = s^2, rs = sr^{-1}\rangle$. Let $U = \langle r\rangle$, then $sU = \{s, sr, sr^2\}$, $Us = \{s, rs, r^2s\}$.

since $sr = r^2s$, $sr^2 = rs$, $\Rightarrow sU = Us$.

Let $H = \langle s\rangle$. Then $rH = \{r, rs = sr^{-1}\}$

$Hr = \{r, sr\}$ $\Rightarrow rH \neq Hr$.

Def: A subgroup K of G is a normal subgroup if $(\forall a\in G, k\in K)$, $aka^{-1}\in K$.

and we write $K \triangleleft G$.

It's easy to check that this definition is equivalent to.

i) $\forall a \in G$, $aK = Ka$, i.e. left coset = right coset.

or ii) $\forall a \in G$, $aKa^{-1} = K$.

Exercise: try to prove the above equivalence.

In previous example, $U \triangleleft D_6$ is normal, but H isn't.

<u>Lemma</u>. 1) Every subgroup of index 2 is normal.

Pf: Let $K \subseteq G$ with index 2, then K has only two possible cosets K, $G-K$. Since $eK = Ke$, we see the nontrivial coset ~~gK and~~ gK and Kg must be $G-K$. hence K is normal.

Clearly, any subgroups of an abelian group is normal.

<u>Prop</u>: Let $f: G \to H$ be a homomorphism, then $\operatorname{Ker} f$ is a normal subgroup of G.

<u>E.g.</u> Let $G = D_8$, then $K = \langle r^2 \rangle \triangleleft G$. In fact, we just need to look at the generator r^2. Note that any $g \in D_8$ has the form r^l or sr^l for some $l \in \mathbb{N}$.

1) $g = sr^l$, $(sr^l)(r^2)(sr^l)^{-1} = sr^l r^2 r^{-l} s^{-1} = sr^2 s^{-1}$
$= sr^2 s = ssr^{-2} = r^2 \in K$.

2) $g = r^l$, $r^l r^2 r^{-l} = r^2 \in K$.

Quotient groups

Let $K \trianglelefteq G$, then the set of (left) coset of K in G is a group under the operation $aK * bK = (ab)K$.

First let's see this operation is well-defined, i.e. for different representative of the coset, we get the same product. Suppose $aK = a'K$, $bK = b'K$. By assumption, we can write $a' = ak_1$, $b' = bk_2$ for some $k_1, k_2 \in K$. Since K is normal, $bk_1b^{-1} \in K$, so $b^{-1}k_1b = k_3 \in K \Rightarrow$ ~~$bk_1 = k_3b$~~. $k_1 b = bk_3$

Now $a'b' = ak_1bk_2 = abk_3k_2 \in abK$.

Hence, $abK = a'b'K$.

<u>Quotient group</u> We define the G/K be the group of cosets of $K \trianglelefteq G$.

E.g. 1) $G = \mathbb{Z}$, $K = n\mathbb{Z}$ $\Rightarrow$ $G/K = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n = \mathbb{Z}/n$

Note, if G is abelian, so is G/K.

2) $K = \langle r \rangle \trianglelefteq D_6$. We have two cosets: K and sK, so $D_6/K = \{e, s\} \cong C_2 \cong \mathbb{Z}/2\mathbb{Z}$.

3). $K = \langle r^2 \rangle \trianglelefteq D_8$, $G/K = \{K, rK = r^3K, sK = sr^2K, srK = sr^3K\}$.

$\Rightarrow$ all elements has order 2, so $G/K = C_2 \times C_2$.

Note: Quotient groups are not <u>subgroups</u>.