Week 2.

Groups

Binary operation: $* : A \times A \to A$.

Def of ~~Gp.~~ Group. G: A group is a set G with a binary operation s.t.

1. $\exists\, e \in G$ such that $\forall\, a \in G$

$$e * a = a * e = a \quad \Rightarrow \text{identity}$$

2. $\forall\, a \in G$, $\exists\, a^{-1} \in G$ with

$$a * a^{-1} = a^{-1} * a = e \quad \Rightarrow \text{existence of inverses.}$$

3. For all $a, b, c \in G$, $(a*b)*c = a*(b*c)$.

Examples:

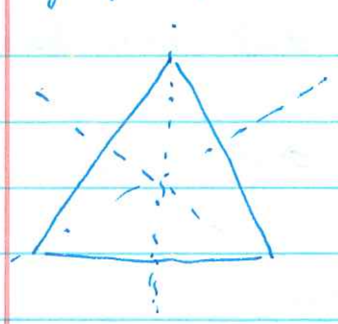
$(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$ $(\mathbb{C}, +)$.

$(\mathbb{Z}/n\mathbb{Z}, +)$ $(\mathbb{Q}^\times, \times)$ note: $\mathbb{Q}^\times = \mathbb{Q} - \{0\}$.

$(\mathbb{C}^*, \times)$ $(\mathbb{R}^*, \times)$.

$(\{-1, 1\}, \times)$.

For a more interesting example, let's look at the ~~symmetry~~ group formed by all symmetries of the triangle (equilateral).



We have 3 rotations $(0°, 120°, 240°)$

3 reflections reflect through axes shown in the picture.

Call rotations by e, r, r^2

~~symmetr~~ reflections by s, rs, r^2s.

inverses $r^{-1} = r^2$ $s^{-1} = s$.

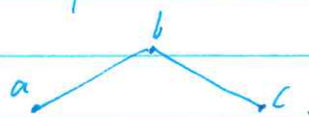
$(rs)^{-1} = rs \Rightarrow rsrs = e. \Rightarrow rs = sr^2.$

$(r^2s)^{-1} = r^2s \Rightarrow r^2sr^2s = e.$

This is an example of Dihedral groups ~~of~~ D_{2n} of order $2n$. D_{2n} ~~is~~ consists of the symmetries of the regular n-gon.

Note all rotations in D_{2n} is a subgroup of D_{2n}, called cyclic groups. A cyclic group is of the form $\{e, a, a^2, \cdots, a^{n-1}\}$ and $a^n = e$. We see $C_n \underset{\text{know}}{\subseteq} D_{2n}$.

Note we have $rs = sr^2 = sr^{-1}$ since if we ~~fix~~ know three consecutive points' position, we know positions of all other points.



Hence we can write D_{2n} as

$$D_{2n} = \{r, s \mid r^n = s^2 = e, \; rs = sr^{-1}\}$$
$$= \{e, r, r^2, \cdots, r^{n-1}; rs, r^2s, r^3s, \cdots, r^{n-1}s\}.$$

Q: Is Dihedral group abelian? i.e. $ab = ba \ \forall \ a, b \in G$?

<u>Symmetric groups.</u>

Let X be a finite set, $|X| = n$. Let $S_n = \mathrm{Sym}\, X$ consists all bijections from X to itself (i.e. permutations).

<u>Exercise</u> Verify S_n is a group.

What is the order of S_n? $\Rightarrow n!$

Notation: 1. two row notation.

i.e. $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \in S_3$.

in general.

if $\sigma : X \to X$ be a permutation, write.

$$\begin{pmatrix} 1 & 2 & \cdots\cdots & n \\ \sigma(1) & \sigma(2) & \cdots\cdots & \sigma(n) \end{pmatrix}.$$

$S_1 = C_1 \quad S_2 = C_2$

What is S_3?

$$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\}$$

Note: $|D_6| = 3$, ~~is~~ are D_6 and S_3 "the same"?

2. Cycle notation.

If a maps sends $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$, we write (123). We leave out the numbers which don't move. i.e. (12) instead of $(12)(3)$.

Ex. $(123)(12) = (13)$

~~Comp~~ composition is from right to left. So

$$1 \xmapsto{(12)} 2 \xmapsto{(123)} 3$$

$$2 \xmapsto{(12)} 1 \xmapsto{(123)} 2 \qquad \Rightarrow \text{We get } (13)(2)$$

$$3 \xmapsto{(12)} 3 \xmapsto{(123)} 1$$

Ex. $(1234)(14) = (234)$.

Q: Is S_3 abelian? How about S_n for $n>3$?

In fact, any non-abelian groups of order 6 is isomorphic to S_3.

More examples:

Consider the group generated by i, $G=\{i,-1,-i,1\}$.

this is a subgroup of the quaternion group.

$Q_8=\{1,-1,i,-i,j,-j,k,-j\}$

such that $i^2=j^2=j^2=-1$

$ij=k \quad ji=-k$

$jk=i \quad kj=-i$

$ki=j \quad ik=-j.$

Note Q_8 is non abelian.

Is Q_8 isomorphic to D_8? No.

Matrix group.

Let ~~$M_n(\mathbb{R})$ be~~ $M_n(\mathbb{R})$ be $n\times n$ matrices with real entries.

Let. $GL_n(\mathbb{R})=\{A\in M_n(\mathbb{R}) \mid \det A\neq 0\}$. (General linear group)

Verify that $GL_n(\mathbb{R})$ is a group.

Consider $SL_n(\mathbb{R})$ to be $SL_n(\mathbb{R})=\{A\in GL_n(\mathbb{R}) \mid \det A=1\}$.

Again, $SL_n(\mathbb{R})$ is a group.