Week 1. §1.2. 1. $F = \{x + y\sqrt{2} \mid x, y \in \mathbb{Q}\}$. We need to show F is a subfield of $\mathbb{C}$.

1) 0 is in F ✓ take $x = y = 0$.

2) 1 is in F ✓ take $x = 1$, $y = 0$

3) if x, y are in F, so is $x + y$

Let $x = a + b\sqrt{2}$ $y = c + d\sqrt{2}$

$\therefore x + y = (a+c) + (b+d)\sqrt{2} \in F$ since $(a+c), (b+d) \in \mathbb{Q}$.

4) if $x = a + b\sqrt{2} \in F$, then $-x = -a - b\sqrt{2} \in F$

5). $x, y \in F$, $xy = (a + b\sqrt{2})(c + d\sqrt{2})$

$= (ab + 2bd) + (ad + bc)\sqrt{2} \in F$

6). $x \neq 0 \in F$, $x^{-1} = \dfrac{1}{a + b\sqrt{2}} = \dfrac{a - \sqrt{2}b}{a^2 - 2b^2}$

$= \dfrac{a}{a^2 - 2b^2} - \dfrac{b}{a^2 - 2b^2}\sqrt{2}$

Note since $x \neq 0$, $a - \sqrt{2}b \neq 0$, $\therefore a^2 - 2b^2 \neq 0$

#3. $\begin{cases} -x_1 + x_2 + 4x_3 = 0 & ① \\ x_1 + 3x_2 + 8x_3 = 0 & ② \\ \frac{1}{2}x_1 + x_2 + \frac{5}{2}x_3 = 0 & ③ \end{cases}$ $\begin{cases} x_1 - x_3 = 0 & ④ \\ x_2 + 3x_3 = 0. & ⑤ \end{cases}$

$\Rightarrow$ ① + ② $\Rightarrow$ $4x_2 + 12x_3 = 0$ $\Rightarrow$ ⑤

② − 3③ $\Rightarrow$ $-\frac{1}{2}x_1 + \frac{1}{2}x_3 = 0$ $\Rightarrow$ ④

$\Leftarrow$ $-x_1 + x_2 + 4x_3 = -$④ + ⑤

$x_1 + 3x_2 + 8x_3 =$ ④ + 3⑤

$\frac{1}{2}x_1 + x_2 + \frac{5}{2}x_3 = \frac{1}{2}$④ + ⑤

#1.3.1. $\begin{cases} (1-i)x_1 - ix_2 = 0 \\ 2x_1 + (1-i)x_2 = 0 \end{cases}$

$\begin{bmatrix} 1-i & -i \\ 2 & 1-i \end{bmatrix}$ row reducing $\Rightarrow \begin{pmatrix} 2 & 1-i \\ 1-i & -i \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 2 & 1-i \\ 0 & 0 \end{pmatrix} \Rightarrow 2x_1 + (1-i)x_2 = 0 \Rightarrow x_1 = \left(\frac{1-i}{2}\right) x_2$

$\therefore$ all solutions have the form $\{(\frac{1}{2}(1-i)z, z) \mid z \in \mathbb{C}\}$