MATH 662 Matrix Analysis   Notes from Hoffman and Kunze 2/e, Chapter 1

Instructions Read the appropriate section in the text first. Then read through the notes and attempt the suggested exercises. The assigned homework is due according to the schedule on the course webpage (`www.math.wcu.edu/~jlawson/teaching/math662/`.

§1.1 Fields Review the nine properties of a field.

Example $\mathbb{Q}$ is the field of rational numbers; $\mathbb{R}$ is the field of real numbers; $\mathbb{C}$ is the field of complex numbers: $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Definition A ***binary operation*** on a nonempty set A is a map $f : A \times A \to A$.

Example Addition in a field is a binary operation: $f(x, y) = x + y$.

Definition For a field $\mathbb{F}$, the ***deleted*** or ***punctured*** field is $\mathbb{F}^* = \mathbb{F} - \{0\}$.

Optional: In abstract algebra, $\mathbb{F}$ is an *abelian group* under the binary operation $+$, $\mathbb{F}^*$ is an abelian group under $\times$, and $\mathbb{F}$ is a *commutative ring with identity* under $+$ and $\times$.

Definition ***Subtraction*** is a binary operation on $\mathbb{F}$ given by $f(x, y) = x + (-y)$. We may denote $f(x, y)$ as $x - y$.

Definition ***Division*** is a map $f : \mathbb{F} \times \mathbb{F}^* \to \mathbb{F}$ given by $f(x, y) = xy^{-1}$. We may denote $f(x, y)$ as $x \div y$.

So division is the composition of the inversion operation $x \mapsto x^{-1}$ and the multiplication operation. Thus a field is an algebraic structure which possesses all four algebraic operations that we used in fourth grade: $+, -, \times, \div$.

Exercise Show that the set of integers $\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ is *not* a field. Show that $\mathbb{Q}(\sqrt{2}) := \{p + q\sqrt{2} \mid p, q \in \mathbb{Q}\}$ is a field.

Definition A ***subfield*** $\mathbb{F}'$ of a field $\mathbb{F}$ is a subset in which $0, 1 \in \mathbb{F}'$, and $\mathbb{F}'$ is closed under addition in $\mathbb{F}$, multiplication in $\mathbb{F}$, and the operations$x \mapsto x^{-1}$ for $x \neq 0$ and $x \mapsto -x$.

Obviously a subfield is a field itself.

Definition If $n \in \mathbb{Z}^+$ is the smallest positive integer such that $n \cdot 1 = 1 + 1 + \cdots + 1$ (n times) $= 0$ then we say that the ***characteristic*** of $\mathbb{F}$ is n, denoted $\operatorname{char} \mathbb{F} = n$. If no such $n \in \mathbb{N}$ exists then we say $\operatorname{char} \mathbb{F} = 0$.

Exercise Show that $\mathbb{Z}_2 := \{0, 1\}$ (where addition is performed modulo 2, e.g., $1 + 1 = 0$) is a field. Show that $\mathbb{Z}_4 := \{0, 1, 2, 3\}$ (where $+$ and $\times$ are performed modulo 4) is *not* a field.

§1.2 Systems of linear equations After reading the text formulate any questions you may have. You may wish to review this subject in an undergraduate linear algebra textbook.

$\boxed{\textbf{HW}\quad 1.2.5,\ 1.2.7}$

§1.3 Matrices and elementary row operations Let $\mathbb{F}^{m\times n}$ denote the set of $m \times n$ matrix with entries in $\mathbb{F}$. Let A_{ij} denote the entry in the ith row and jth column of A. In $\mathbb{F}^{n\times n}$, I_n (or just I when the size of the square matrix is obvious or generic) denotes the $n \times n$ identity matrix. A succinct way to describe I is with the ***Kronecker delta***

$$I_{ij} = \delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} .$$

In the land of abstract algebra $\mathbb{F}^{n\times n}$ is a ring with identity under matrix multiplication and addition, but it fails to be a field.

Exercise Find a field property that $\mathbb{F}^{2\times 2}$ violates by supplying a counterexample to that property.

In linear algebra, fields are appropriate sets to serve as ***scalars***. Thus your counterexample illustrates that square matrices serve poorly as scalars. Read this section paying attention especially to the three legal elementary row operations and ***row-reduced matrices***. An important fact from this section is that two $m \times n$ ***row-equivalent matrices*** have exactly the same solutions $X \in \mathbb{F}^m$ to the systems $AX = 0$ and $BX = 0$ where $0 \in \mathbb{F}^n$. Again, refer to an undergraduate textbook if you desire.

$\boxed{\textbf{HW}\quad 1.3.1,\ 1.3.3}$

§1.4 Row-reduced echelon matrices Review the definition of a ***row-reduced echelon matrix***, especially the formal definition on p. 12.

Exercise Find example of a row-reduced matrix that is *not* a row-reduced echelon matrix.

Read Theorems 6 and 7, p. 13. Under the conditions of Theorem 6, the system is ***underdetermined***. The biconditional statement in Theorem 7 is equivalent to saying that $\det A \neq 0$. Also review solving systems $AX = Y$ by augmented matrices.

$\boxed{\textbf{HW}\quad 1.4.2,\ 1.4.3,\ 1.4.5}$

§1.5 Matrix multiplication Review matrix multiplication, paying attention to the indices in the formal notation. For $C = AB$, $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times p}$

$$C_{ij} = \sum_{r=1}^{n} A_{ir}B_{rj}\,, \quad i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, p\,.$$

Exercise Write out the formula in indices for $D = ABC$, where A and B are as above, and $C \in \mathbb{F}^{p\times q}$.

Multiplication of matrices is associate (Theorem 8) but not commutative. Review Theorem 9 and its Corollary.

HW 1.5.3, 1.5.8

§1.6 Invertible matrices Note that I_n serves as both a left and a right identity in the set of $n \times n$ matrices. Namely, $\forall A \in \mathbb{F}^{n\times n}, I_n A = A = AI_n$.

Definition Let $A, B, C \in \mathbb{F}^{n\times n}$. If $BA = I_n$ then B is a ***left inverse*** of A. If $AC = I_n$ then C is a ***right inverse*** of A.

By the Lemma on p. 22, $C = B$. Read the proof. Thus, we have earned the privilege to denote the unique inverse by $C = B =: A^{-1}$.

Exercise Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$. Find $B \in \mathbb{R}^{2\times 3}$ such that $BA = I_2$. Is B unique? Show that there does not exist any C such that $AC = I_3$. Thus the left and right inverses may differ for *nonsquare* matrices.

Review Theorems 12 and 13 and their Corollaries.

HW 1.6.3, 1.6.7, 1.6.9

End of chapter.