Instructions Read the appropriate section in the text first. Then read through the notes and attempt the suggested exercises. The assigned homework is due according to the schedule on the course webpage (`www.math.wcu.edu/~jlawson/teaching/math662/`.

§2.1 Vector spaces Review the four properties of a vector space.

1. We have a field $\mathbb{F}$ of ***scalars***.
2. We have a set V of objects called ***vectors***. Typically V must not be the empty set $\emptyset$.
3. The set V has a commutative and associative binary operator $+$, an additive identity and an additive inverse (and thus we get subtraction for free). In abstract algebra, we say that V is an abelian group under $+$.
4. ***Scalar multiplication*** is a a *new* algebraic structure that describes the rules of engagement for $\mathbb{F}$ and V. For all $c_1, c_2 \in \mathbb{F}$ and $\alpha_1, \alpha_2 \in V$:
   (a) The identity $1 \in \mathbb{F}$ serves as the scalar multiplicative identity: $1\,\alpha = \alpha$.
   (b) "Associativity:" $c_1(c_2\alpha) = (c_1c_2)\alpha)$. (Thus $c_1c_2\alpha)$ is disambiguous.)
   (c) Scalar multiplication distributes over addition in V: $c_1(\alpha_1 + \alpha_2) = c_1\alpha_1 + c_1\alpha_2$.
   (d) Scalar multiplication distributes over addition in $\mathbb{F}$: $(c_1 + c_2)\alpha_1 = c_1\alpha_1 + c_2\alpha_1$.

Exercise Read Examples 1–5 and verify that they are vector spaces.

When at risk of confusion, state the field you are using.

Exercise Describe the elements of the set

$$\left\{ c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ \pi \end{bmatrix} \;\middle|\; c_1, c_2 \in \mathbb{F} \right\}$$

for $\mathbb{F} = \mathbb{R}$ and for $\mathbb{F} = \mathbb{Q}$. Compare the two sets.

Read the Lemma following the examples and the proof: If $c \in \mathbb{F}$, $\alpha \in V$, and $c\alpha = 0$ then $c = 0$ or $\alpha = 0$. Observe that there are two meanings for "0:" If $c = 0$ then we mean $0 \in f$ to be a scalar, but if $\alpha = 0$ then we intend $0 \in V$ to be the ***zero vector***. Gone is the vector notation from our days of multivariable calculus (like putting an arrow over a vector or writing it in boldface or underline). Perhaps as a bit of mathematical snobbery (or laziness), we distinguish scalars from vectors mostly by context.

Definition Given a *finite* collection of vectors $\{\alpha_i\}_{i=1}^n \subset V$ the set of ***linear combinations*** of that collection is

$$\left\{ \sum_{i=1}^n \mathbb{C}_i\alpha_i \;\middle|\; c_i \in \mathbb{F} \right\}.$$

Exercise Can $\begin{bmatrix} 0 \\ e \end{bmatrix}$ be written as a linear combination of the set $\left\{ \begin{bmatrix} 0 \\ e \end{bmatrix}, \begin{bmatrix} 0 \\ e \end{bmatrix} \right\}$ if $\mathbb{F} = \mathbb{R}$? If $\mathbb{F} = \mathbb{Q}$?

Read about the geometry of n-tuples represented as vectors in $\mathbb{F}^n$. Vectors in the cartesian plane $\mathbb{R}^2$ are identified with points $P(x, y)$ by constructing the head-to-tail vector $\vec{OP}$ where 0 is the origin. (Thus the zero vector $0 \in \mathbb{R}^2$ is identified with the origin as a point $(0, 0)$. This gives us the familiar picture from vector calculus.

HW 2.1.3, 2.1.4, 2.1.7

§2.2 Subspaces For subsets we may write $W \subset V$ or $W \subseteq V$. The former usually indicates a *proper* subset.

Definition Let V be a vector space over field $\mathbb{F}$. A subset $W \subset V$ is a ***subspace*** if W is also a vector space over $\mathbb{F}$ using the addition and scalar multiplication inherited from V.

Some folks use the notation $W < V$ or $W \leq V$ to indicate that W is also a subspace (the former a proper subspace) of V. Observe that $\{0\} < V$ and of course $V \leq V$.

Read the Theorem about necessary and sufficient conditions for $W \subset V$ to be a subspace, and the proof. Observe that all properties of a vector space are inherited from V. To verify that W is a subspace we need only to check that W is closed under addition and scalar multiplication. The theorem provides an efficient one-step test for closure. Review the examples of subspaces in the text.

Exercise Show that the set of *antisymmetric* matrices ($A_{ij} = -A_{ji}$) over $\mathbb{F}$ do *not* form a subspace of $\mathbb{F}^{n\times n}$

Review the properties of matrix multiplication. Observe that scalars "pop out" in scalar multiplication: $A(dB + C) = d(AB) + AC$. The text has a formal proof using indices.

Study Theorem 2 and its proof. Observe that this applies to an *arbitrary* collection, not just a finite collection.

Exercise Find two subspaces W_1 and W_2 of $\mathbb{R}^2$ so that $W_1 \cup W_2$ is *not* a subspace.

Definition Let $S = \{\alpha_i\}_{i=1}^n$, where $\alpha_i \in V$. Let $\{W_a\}_{a\in A}$ be the set of all subspaces W_a of V such that $S \subset W_a$. (The set A is the indexing set.) Then the ***subspace spanned by***

the vectors in S is defined to be

$$\operatorname{span} S := \bigcap_{a \in A} W_a \,.$$

Theorem 3

$$\operatorname{span} S = \left\{ \sum_{i=1}^{n} c_i \alpha_i \,\middle|\, c_i \in \mathbb{F} \right\}$$

Read the proof. We require S to be nonempty.

HW 2.2.1, 2.2.3

Definition Let $S = \{\alpha_i\}_{i=1}^n$, where $\alpha_i \in V$. Let $\{W_a\}_{a \in A}$ be the set of all subspaces W_a of V such that $S \subset W_a$. (The set A is the indexing set.) Then the ***subspace spanned by the vectors in*** S is defined to be

$$\operatorname{span} S := \bigcap_{a \in A} W_a \,.$$

Read the definition of a (finite) ***sum*** of subsets of V. If all of the subsets are also subspaces then the sum is a subspace. (Why?) Read the examples, especially the example involving the ***rowspace*** of a matrix.

Exercise Convince yourself that $\operatorname{span} \emptyset = \{0\}$, the zero vector in V.

Lemma 1 *Let* $\{W_i\}_{i=1}^n$ *be a finite collection of subspaces of* V*. Then*

$$\bigcup_{i=1}^{n} W_i \subset \sum_{i=1}^{n} W_i$$

Exercise Prove.

HW 2.2.5, 2.2.6a

Continue with §2.3 now.