MATH 601 ALGEBRAIC TOPOLOGY HW 2
SELECTED SOLUTIONS SKETCH/HINT

QINGYUN ZENG

1. Problem 6

It suffices to prove the following claim.

Lemma 1.1. *Let $p : \tilde{X} \to X$ be a covering map. If X is a path connected space, $x_0, x_1 \in X$, then there is a bijection $p^{-1}(x_0) \to p^{-1}(x_1)$.*

Proof. Let $\gamma : x_0 \rightsquigarrow x_1$ be a path. We want to use this to construct a bijection between each preimage of x_0 and each preimage of x_1. The obvious thing to do is to use lifts of the path γ.

Define a map $f_\gamma : p^{-1}(x_0) \to p^{-1}(x_1)$ that sends $\tilde{x}_0$ to the end point of the unique lift of γ at $\tilde{x}_0$.

The inverse map is obtained by replacing γ with γ^{-1}, i.e. $f_{\gamma^{-1}}$. To show this is an inverse, suppose we have some lift $\tilde{\gamma} : \tilde{x}_0 \rightsquigarrow \tilde{x}_1$, so that $f_\gamma(\tilde{x}_0) = \tilde{x}_1$. Now notice that $\tilde{\gamma}^{-1}$ is a lift of γ^{-1} starting at $\tilde{x}_1$ and ending at $\tilde{x}_0$. So $f_{\gamma^{-1}}(\tilde{x}_1) = \tilde{x}_0$. So $f_{\gamma^{-1}}$ is an inverse to f_γ, and hence f_γ is bijective. $\square$

2. Problem 7 & 8

We begin with the following lemma:

Lemma 2.1. *Let $p : \tilde{X} \to X$ be a covering map, $f : Y \to X$ be a map, and $\tilde{f}_1, \tilde{f}_2$ be both lifts of f. Then*

$$S = \{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\}$$

is both open and closed. In particular, if Y is connected, $\tilde{f}_1$ and $\tilde{f}_2$ agree either everywhere or nowhere.

This is sort of a "uniqueness statement" for a lift. If we know a point in the lift, then we know the whole path. This is since once we've decided our starting point, i.e. which "copy" of X we work in, the rest of $\tilde{f}$ has to follow what f does.

Proof. First we show it is open. Let y be such that $\tilde{f}_1(y) = \tilde{f}_2(y)$. Then there is an evenly covered open neighbourhood $U \subseteq X$ of $f(y)$. Let $\tilde{U}$ be such that $\tilde{f}_1(y) \in \tilde{U}$, $p(\tilde{U}) = U$ and $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism. Let $V = \tilde{f}_1^{-1}(\tilde{U}) \cap \tilde{f}_2^{-1}(\tilde{U})$. We will show that $\tilde{f}_1 = \tilde{f}_2$ on V.

Indeed, by construction

$$p|_{\tilde{U}} \circ \tilde{f}_1|_V = p|_{\tilde{U}} \circ \tilde{f}_2|_V.$$

Since $p|_{\tilde{U}}$ is a homeomorphism, it follows that

$$\tilde{f}_1|_V = \tilde{f}_2|_V.$$

Now we show S is closed. Suppose not. Then there is some $y \in \bar{S} \setminus S$. So $\tilde{f}_1(y) \neq \tilde{f}_2(y)$. Let U be an evenly covered neighbourhood of $f(y)$. Let $p^{-1}(U) = \coprod U_\alpha$. Let $\tilde{f}_1(y) \in U_\beta$ and $\tilde{f}_2(y) \in U_\gamma$, where $\beta \neq \gamma$. Then $V = \tilde{f}_1^{-1}(U_\beta) \cap \tilde{f}_2^{-1}(U_\gamma)$ is an open neighbourhood of y, and hence intersects S by definition of closure. So there is some $x \in V$ such that $\tilde{f}_1(x) = \tilde{f}_2(x)$. But $\tilde{f}_1(x) \in U_\beta$ and

$\tilde{f}_2(x) \in U_\gamma$, and hence U_β and U_γ have a non-trivial intersection. This is a contradiction. So S is closed. □

We just had a uniqueness statement. How about existence? Given a map, is there guarantee that we can lift it to something? Moreover, if I have fixed a "copy" of X I like, can I also lift my map to that copy? We will later come up with a general criterion for when lifts exist. However, it turns out homotopies can always be lifted.

Lemma 2.2 (Homotopy lifting lemma)**.** *Let $p : \tilde{X} \to X$ be a covering space, $H : Y \times I \to X$ be a homotopy from f_0 to f_1. Let $\tilde{f}_0$ be a lift of f_0. Then there exists a* unique *homotopy $\tilde{H} : Y \times I \to \tilde{X}$ such that*

1. *$\tilde{H}(\cdot, 0) = \tilde{f}_0$; and*
2. *$\tilde{H}$ is a lift of H, i.e. $p \circ \tilde{H} = H$.*

This lemma might be difficult to comprehend at first. We can look at the special case where $Y = *$. Then a homotopy is just a path. So the lemma specializes to

Lemma 2.3 (Path lifting lemma)**.** *Let $p : \tilde{X} \to X$ be a covering space, $\gamma : I \to X$ a path, and $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0 = \gamma(0)$. Then there exists a* unique *path $\tilde{\gamma} : I \to \tilde{X}$ such that*

1. *$\tilde{\gamma}(0) = \tilde{x}_0$; and*
2. *$\tilde{\gamma}$ is a lift of γ, i.e. $p \circ \tilde{\gamma} = \gamma$.*

This is exactly the picture we were drawing before. We just have to start at a point $\tilde{x}_0$, and then everything is determined because locally, everything upstairs in $\tilde{X}$ is just like X. Note that we have already proved uniqueness. So we just need to prove existence.

In theory, it makes sense to prove homotopy lifting, and path lifting comes immediately as a corollary. However, the proof of homotopy lifting is big and scary. So instead, we will prove path lifting, which is something we can more easily visualize and understand, and then use that to prove homotopy lifting.

Proof. Let

$$S = \{s \in I : \tilde{\gamma} \text{ exists on } [0, s] \subseteq I\}.$$

Observe that

1. $0 \in S$.
2. S is open. If $s \in S$ and $\tilde{\gamma}(s) \in V_\beta \subseteq p^{-1}(U)$, we can define $\tilde{\gamma}$ on some small neighbourhood of s by
$$\tilde{\gamma}(t) = (p|_{V_\beta})^{-1} \circ \gamma(t)$$
3. S is closed. If $s \notin S$, then pick an evenly covered neighbourhood U of $\gamma(s)$. Suppose $\gamma((s - \varepsilon, s)) \subseteq U$. So $s - \frac{\varepsilon}{2} \notin S$. So $(s - \frac{\varepsilon}{2}, 1] \cap S = \emptyset$. So S is closed.

Since S is both open and closed, and is non-empty, we have $S = I$. So $\tilde{\gamma}$ exists. □

How can we promote this to a proof of the homotopy lifting lemma? At every point $y \in Y$, we know what to do, since we have path lifting. So $\tilde{H}(y, \cdot)$ is defined. So the thing we have to do is to show that this is continuous. Steps of the proof are

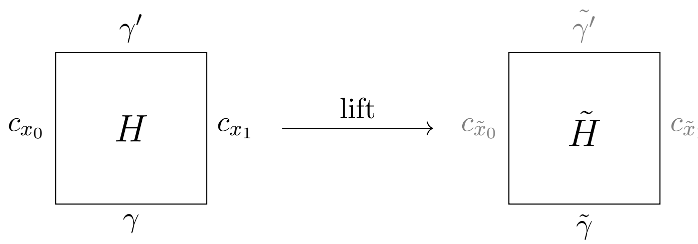
1. Use compactness of I to argue that the proof of path lifting works on small neighbourhoods in Y.
2. For each y, we pick an open neighbourhood U of y, and define a good path lifting on $U \times I$.
3. By uniqueness of lifts, these path liftings agree when they overlap. So we have one big continuous lifting.

3. Problem 9

This problem is esstentially a corollary of previous two.

Suppose $\gamma, \gamma' : I \to X$ are paths $x_0 \rightsquigarrow x_1$ and $\tilde{\gamma}, \tilde{\gamma}' : I \to \tilde{X}$ are lifts of γ and γ' respectively, both starting at $\tilde{x}_0 \in p^{-1}(x_0)$. We want to show that if $\gamma \simeq \gamma'$ as *paths*, then $\tilde{\gamma}$ and $\tilde{\gamma}'$ are homotopic as paths. In particular, $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$. Note that if we cover the words "as paths" and just talk about homotopies, then this is just the homotopy lifting lemma. So we can view this as a stronger form of the homotopy lifting lemma.

Proof. The homotopy lifting lemma gives us an $\tilde{H}$, a lift of H with $\tilde{H}(\cdot, 0) = \tilde{\gamma}$.



In this diagram, we by assumption know the bottom of the $\tilde{H}$ square is $\tilde{\gamma}$. To show that this is a path homotopy from $\tilde{\gamma}$ to $\tilde{\gamma}'$, we need to show that the other edges are $c_{\tilde{x}_0}$, $c_{\tilde{x}_1}$ and $\tilde{\gamma}'$ respectively.

Now $\tilde{H}(\cdot, 1)$ is a lift of $H(\cdot, 1) = \gamma'$, starting at $\tilde{x}_0$. Since lifts are unique, we must have $\tilde{H}(\cdot, 1) = \tilde{\gamma}'$. So this is indeed a homotopy between $\tilde{\gamma}$ and $\tilde{\gamma}'$. Now we need to check that this is a homotopy of paths.

We know that $\tilde{H}(0, \cdot)$ is a lift of $H(0, \cdot) = c_{x_0}$. We are aware of one lift of c_{x_0}, namely $c_{\tilde{x}_0}$. By uniqueness of lifts, we must have $\tilde{H}(0, \cdot) = c_{\tilde{x}_0}$. Similarly, $\tilde{H}(1, \cdot) = c_{\tilde{x}_1}$. So this is a homotopy of paths. □

4. Problem 11

Let $p : \mathbb{R} \to S^1$ be the universal covering map of S^1.

By problem 11, there is a bijection $\pi_1(S^1, 1) \to p^{-1}(1) = \mathbb{Z}$, hence $\pi_1(S^1, 1)$ is countably infinite. Next, we want to show p is a group isomorphism.

We know it is a bijection. So we need to check it is a group homomorphism. The idea is to write down representatives for what we think the elements should be.

Let $\tilde{u}_n : I \to \mathbb{R}$ be defined by $t \mapsto nt$, and let $u_n = p \circ \tilde{u}_n$. Since $\mathbb{R}$ is simply connected, there is a unique homotopy class between any two points. So for any $[\gamma] \in \pi_1(S^1, 1)$, if $\tilde{\gamma}$ is the lift to $\mathbb{R}$ at 0 and $\tilde{\gamma}(1) = n$, then $\tilde{\gamma} \simeq \tilde{u_n}$ as paths. So $[\gamma] = [u_n]$.

To show that this has the right group operation, we can easily see that $\widetilde{u_m \cdot u_n} = \tilde{u}_{m+n}$, since we are just moving by $n + m$ in both cases. Therefore

$$\ell([u_m][u_n]) = \ell([u_m \cdot u_m]) = m + n = \ell([u_{m+n}]).$$

So ℓ is a group isomorphism.

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Current address: Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104
E-mail address: qze@math.upenn.edu