MATH 601 ALGEBRAIC TOPOLOGY HW 3
SELECTED SOLUTIONS SKETCH/HINT

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1. Problem 7

Recall that $\mathbb{R}P^n \cong S^n/\{\pm \mathrm{Id}\}$, and the quotient map $S^n \to \mathbb{R}P^n$ is a covering map. Now that we have proved that S^n is simply connected, we know that S^n is a universal cover of $\mathbb{R}P^n$.

For any $x_0 \in \mathbb{R}P^n$, we have a bijection

$$\pi_1(\mathbb{R}P^n, x_0) \leftrightarrow p^{-1}(x_0).$$

Since $p^{-1}(x_0)$ has two elements by definition, we know that $|\pi_1(\mathbb{R}P^n, x_0)| = 2$. So $\pi_1(\mathbb{R}P^n, x_0) = \mathbb{Z}/2$.

2. Problem 19

We want to prove the *Brouwer's fixed point theorem* using properties of retraction.

Theorem 2.1 (Brouwer's fixed point theorem)**.** *If $f : D^2 \to D^2$ is a continuous map, then there is some $x \in D^2$ such that $f(x) = x$.*

Proof. Suppose not. So $x \neq f(x)$ for all $x \in D^2$.

We define $g : D^2 \to S^1$ as in the picture above. Then we know that g is continuous and g is a retraction from D^2 onto S^1. In other words, the following composition is the identity:

$$\begin{tikzcd} S^1 \arrow[r, hook, "\iota"] \arrow[rr, bend right, "\mathrm{Id}_{S^1}"'] & D^2 \arrow[r, "g"] & S^1 \end{tikzcd}$$

Then this induces a homomorphism of groups whose composition is the identity:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, hook, "\iota_*"] \arrow[rr, bend right, "\mathrm{Id}_{\mathbb{Z}}"'] & \{0\} \arrow[r, "g_*"] & \mathbb{Z} \end{tikzcd}$$

But this is clearly nonsense! So we must have had a fixed point. ☐

A generalization is the following:

Theorem 2.2. *Every continuous function from a compact convex set in $\mathbb{R}^n$ to itself has at least one fixed point.*

A natural question will be how could we generalize this to infinite dimensional spaces? For example, Hilbert space or Frechét space.

For those who are interested in homotopy type theory, there is a paper arXiv:1509.07584 by Michael Shulman generalizes Brouwer's fixed point theorem in real-cohesive homotopy type theory.

References

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