MATH 601 ALGEBRAIC TOPOLOGY HW 4
SELECTED SOLUTIONS SKETCH/HINT

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1. The Seifert-van Kampen theorem

1.1. **A refinement of the Seifert-van Kampen theorem.** We are going to make a refinement of the theorem so that we don't have to worry about that openness problem. We first start with a definition.

Definition 1.1 (Neighbourhood deformation retract)**.** A subset $A \subseteq X$ is a *neighbourhood deformation retract* if there is an open set $A \subset U \subset X$ such that A is a strong deformation retract of U, i.e. there exists a retraction $r : U \to A$ and $r \simeq \mathrm{Id}_U \, rel A$.

This is something that is true most of the time, in sufficiently sane spaces.

Example 1.2. If Y is a subcomplex of a cell complex, then Y is a neighbourhood deformation retract.

Theorem 1.3. *Let X be a space, $A, B \subseteq X$ closed subspaces. Suppose that A, B and $A \cap B$ are path connected, and $A \cap B$ is a neighbourhood deformation retract of A and B. Then for any $x_0 \in A \cap B$.*

$$\pi_1(X, x_0) = \pi_1(A, x_0) \underset{\pi_1(A\cap B, x_0)}{*} \pi_1(B, x_0).$$

This is just like Seifert-van Kampen theorem, but usually easier to apply, since we no longer have to "fatten up" our A and B to make them open.

If you know some sheaf theory, then what Seifert-van Kampen theorem really says is that the fundamental groupoid $\Pi_1(X)$ is a cosheaf on X. Here $\Pi_1(X)$ is a category with object pints in X and morphisms as homotopy classes of path in X, which can be regard as a global version of $\pi_1(X)$.

1.2. **A generalization of the Seifert-van Kampen theorem.** Here's a generalization of the Seifert-van Kampen theorem by Jacob Lurie, which describes the entire weak homotopy type of X in terms of any stoficiently nice covering of X by open sets.

Theorem 1.4. *Let X be a topological space, let $U(X)$ denote the collection of all open subsets of X (partially ordered by inclusion). Let C be a small category and let $\chi : \mathsf{C} \to U(X)$ be a functor. For every $x \in X$, let C_x denote the full subcategory of C spanned by those objects $C \in \mathsf{C}$ such that $x \in \chi(C)$. Assume that χ satisfies the following condition:*

(1) *For every point x, the simplicial set $N(\mathsf{C}_x)$ is weakly contractible.*

Then the canonical map $\lim_{C\in\mathsf{C}} Sing(\chi(C)) \to Sing(X)$ exhibits the simplicial set $Sing(X)$ as a homotopy colimit of the diagram $\{Sing(\chi(C))\}_{C\in\mathsf{C}}$.

Proof. See Higher algebra A.3.1. □

2. Problem 4

Let $G = (a, b : a^4 = 1, b^2 = 1, bab^{-1} = a^{-1})$. Since $bab^{-1} = a^{-1}$, $ba = a^{-1}b$. From this,

$$ba^2 = (ba)a = (a^{-1}b)a = a^{-1}(ba) = a^{-2}b. \tag{2.1}$$

Similarly we can get $ba^3 = a^{-3}b$. Note that $a^{-n}b = a^{4-n}b$ for $1 \leq n \leq 3$. Hence we see that $G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ since all other elements can be reduced by our observation. Now we show that G is isomorphic to D_4. Let $\phi : G \to D_4$ be a homomorphism defined by $\phi(a) = a'$, $\phi(b) = b'$, where a' is a generator of rotation and b' is a generator of reflexion on D_4. Note that $D_4 = \{e, a', a'^2, a'^3, b', a'b, a'^2b, a'^3b\}$ and satisfies the relation $ba^n = a^{-n}b$. It's easy to see that ϕ is surjective. Since $|G| = |D^4|$, ϕ is injective, hence an isomorphism.

3. Problem 6

Let $m = xyx$ and $n = xy$. We claim that $\{m, n\}$ generates H as well. In fact $m^{-1}n^2 = y$ and $ny^{-1} = x$ and the result follows. Define a homomorphism $\phi : G \to H$ by letting $\phi(a) = m$ and $\phi(b) = n$. Since m and n generate H, ϕ is onto. Now let $\phi(z) = e$. Note that $e \in H$ is generated by $xyxy^{-1}x^{-1}y^{-1}$, we can write it as

$$(xyx)(y^{-1}x^{-1})y^{-1} = mn^{-1}(n^{-2}m) = mn^{-3}m, \tag{3.1}$$

since

$$y^{-1} = (m^{-1}n^2)^{-1} = n^{-2}m. \tag{3.2}$$

Hence $\ker(\phi)$ is generated by $ab^{-3}a$. In fact,

$$ab^{-3} = ab^{-3}e = ab^{-3}(b^3a^{-2}) = a^{-1}, \tag{3.3}$$

so $ab^{-3}a = e$. Therefore, the $\ker(\phi)$ is trivial and hence ϕ is an isomorphism.

4. Problem 14

Consider n-sphere $S^n = \{\mathbf{v} \in \mathbb{R}^{n+1} : |\mathbf{v}| = 1\}$ for $n \geq 2$. We want to find $\pi_1(S^n)$.

The idea is to write S^n as a union of two open sets. We let $n = \mathbf{e}_1 \in S^n \subseteq \mathbb{R}^{n+1}$ be the North pole, and $s = -\mathbf{e}_1$ be the South pole. We let $A = S^n \setminus \{n\}$, and $B = S^n \setminus \{s\}$. By stereographic projection, we know that $A, B \cong \mathbb{R}^n$. The hard part is to understand the intersection.

To do so, we can draw a cylinder $S^{n-1} \times (-1, 1)$, and project our $A \cap B$ onto the cylinder. We can similarly project the cylinder onto $A \cap B$. So $A \cap B \cong S^{n-1} \times (-1, 1) \simeq S^{n-1}$, since $(-1, 1)$ is contractible.

We can now apply the Seifert-van Kampen theorem. Note that this works only if S^{n-1} is path-connected, i.e. $n \geq 2$. Then this tells us that

$$\pi_1(S^n) \cong \pi_1(\mathbb{R}^n) \underset{\pi_1(S^{n-1})}{*} \pi_1(\mathbb{R}^n) \cong 1 \underset{\pi_1(S^{n-1})}{*} 1$$

It is easy to see this is the trivial group. We can see this directly form the universal property of the amalgamated free product, or note that it is the quotient of $1 * 1$, which is 1.

So for $n \geq 2$, $\pi_1(S^n) \cong 1$.

5. Problem 15

Suppose we take the wedge sum of two circles $S^1 \wedge S^1$. We would like to pick A, B to be each of the circles, but we cannot, since A and B have to be open. Notice that both A and B retract to the circle. So $\pi_1(A) \cong \pi_1(B) \cong \mathbb{Z}$, while $A \cap B$ is a cross, which retracts to a point. So $\pi_1(A \cap B) = 1$.

Hence by the Seifert-van Kampen theorem, we get

$$\pi_1(S^1 \wedge S^1, x_0) = \pi_1(A, x_0) \underset{\pi_1(A\cap B, x_0)}{*} \pi_1(B, x_0) \cong \mathbb{Z} \underset{1}{*} \mathbb{Z} \cong \mathbb{Z} * \mathbb{Z} \cong F_2,$$

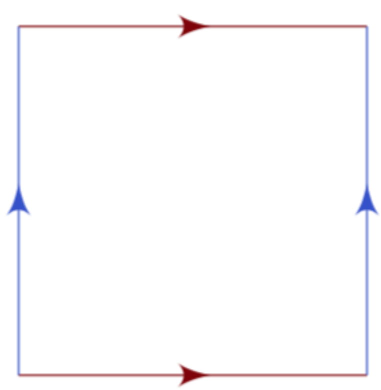
where F_2 is just $F(S)$ for $|S| = 2$. We can see that $\mathbb{Z} * \mathbb{Z} \cong F_2$ by showing that they satisfy the same universal property.

Note that we had already figured this out when we studied the free group, where we realized F_2 is the fundamental group of this thing.

More generally, as long as x_0, y_0 in X and Y are "reasonable", $\pi_1(X \wedge Y) \cong \pi_1(X) * \pi_1(Y)$.

6. Problem 16

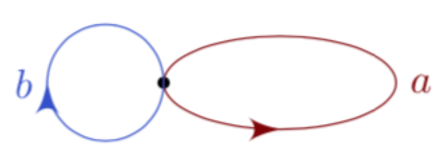
Let X be the 2-torus. Possibly, our favorite picture of the torus is (not a doughnut):



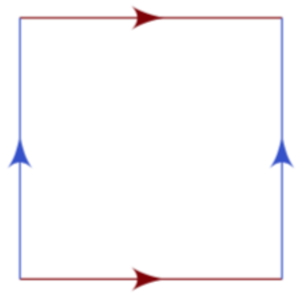
This is already a description of the torus as a cell complex!

We start with our zero complex $X^{(0)}$ which is a point $\bullet$.

We then add our 1-cells to get $X^{(1)}$:



We now glue our square to the cell complex to get $X = X^{(2)}$:



matching up the colors and directions of arrows.

So we have our torus as a cell complex. What is its fundamental group? There are many ways we can do this computation, but this time we want to do it as a cell complex.

We start with $X^{(0)}$. This is a single point. So its fundamental group is $\pi_1(X^{(0)}) = 1$.

When we add our two 1-cells, we get $\pi_1(X^{(1)}) = F_2/ < a, b >$.

Finally, to get $\pi_1(X)$, we have to quotient out by the boundary of our square, which is just $aba^{-1}b^{-1}$. So we have

$$\pi_1(X^{(2)}) = F_2/ < aba^{-1}b^{-1} >=< a, b \mid aba^{-1}b^{-1} > \mathbb{Z}^2.$$

We have the last congruence since we have two generators, and then we make them commute by quotienting the commutator out.

References

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