MATH 601 ALGEBRAIC TOPOLOGY HW 5 SELECTED SOLUTIONS SKETCH/HINT

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1. Covering space and étalé space

An **étalé space** (or ***étalé map***) over B is an object $p : E \to B$ in **Top**$/B$ such that p is a *local homeomorphism*: that is, for every $e \in E$, there is an open set $U \ni e$ such that the image $p(U)$ is open in B and the restriction of p to U is a homeomorphism $p|_U : U \to p(U)$.

The set $E_x = p^{-1}(x)$ where $x \in B$ is called the **stalk** of p over x.

The underlying set of the *total space* E is the union of its stalks.

Every covering space (even in the more general sense not requiring any connectedness axiom) is an étalé space, but not vice versa.

2. Galois correspondance

In the lecture, we established a correspondence between covering spaces and fundamental groups.

We can have the following table of correspondences:

Covering spaces		**Fundamental group**
(Based) covering spaces	$\longleftrightarrow$	Subgroups of π_1
Number of sheets	$\longleftrightarrow$	Index
Universal covers	$\longleftrightarrow$	Trivial subgroup

In fact, the role of fundamental groups is similar to the role of Galois groups in number theory/algebraic geometry.

Grothendieck defined the fundamental group of a scheme, which is essentially the fundamental group in étalé homotopy.

Let S be a connected scheme. A finite étalé cover of S is a finite flat surjection $X \to S$ such that each fiber at a point $s \in S$ is the spectrum of a finite étalé algebra over the local ring at s. Fix a geometric point $\overline{s} : Spec(\Omega) \to \Omega$.

For a finite étale cover, $X \to S$, we consider the geometric fiber $X \times_S Spec(\Omega)$, over $\overline{s}$, and denote by $Fib_{\overline{s}}(X)$ its underlying set. This gives a set-valued functor on the category of finite étale covers of X.

The *étale fundamental group*, $\pi_1(S, \overline{s})$ is defined to be the automorphism group of this functor.

If $R = k$ is a field, then the étale fundamental group recovers Galois group.

3. Problem 2

There are 7 3-sheet covering of $S^1 \vee S^1$ as shown in the picture below.

4. Problem 4

Let $p : E \to B$ be a covering map, where E and B are path connected spaces. Let $b_0 \in B$, and $e_0 \in p^{-1}b_0$. Clearly, p_* sends $p_*(\pi_1(E, e_0))$ into a subgroup of $\pi_1(B, b_0)$. Let e_1 be another point in $p^{-1}(b_0)$, we need show $p_*(\pi_1(E, e_0))$ and $p_*(\pi_1(E, e_1))$ are conjugate to each other. Since E is path

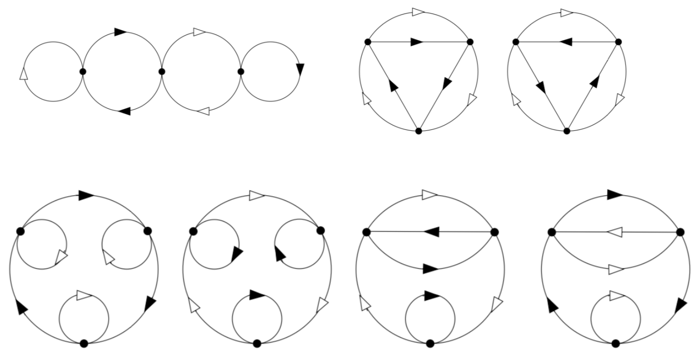

FIGURE 1. All 3-sheet covering of $S^1 \vee S^1$

connected, let $\tilde{c}$ be a path in E connecting e_0 and e_1. Clearly $c = \tilde{c} \circ p$ is a loop in B based at b_0. Let $[\tilde{f}] \in \pi_1(E, e_0)$, then $c^{-1} \cdot \tilde{f} \cdot c$ is a loop based at e_1, so we have $[\tilde{c}^{-1}] \cdot [\tilde{f}] \cdot [\tilde{c}] \in \pi_1(E, e_1)$. Now,

$$p_*([\tilde{c}^{-1}] \cdot [\tilde{f}] \cdot [\tilde{c}]) = p_*[\tilde{c}^{-1} \cdot \tilde{f} \cdot \tilde{c}] \tag{4.1}$$

$$= [p \circ (\tilde{c}^{-1} \cdot \tilde{f} \cdot \tilde{c})] \tag{4.2}$$

$$= [(p \circ \tilde{c}^{-1}) \cdot (p \circ \tilde{f}) \cdot (p \circ \tilde{c})] \tag{4.3}$$

$$= [c^{-1} \cdot f \cdot c] = [c^{-1}] \cdot [f] \cdot [c] \tag{4.4}$$

Since $[f] = [(p \circ \tilde{f}] = p_*[\tilde{f}]$, we see that $\Phi_c(f) = [c^{-1}] \cdot [f] \cdot [c]$ maps $p_*(\pi_1(E, e_0))$ to $p_*(\pi_1(E, e_1))$ and they are conjugate in $\pi_1(B, b_0)$.

Conversely, let G be a subgroup of $\pi_1(B, b_0)$ and is conjugate to $p_*(\pi_1(E, e_0))$. Hence there is some $[c] \in \pi_1(B, b_0)$ such that $H = \Phi_c(p_*(\pi_1(E, e_0)))$. Let $\tilde{c}$ be the unique lift of c starting from $e_0 \in E$ and ends at $c(1) = e_1$. Then we have $p_*(\pi_1(E, e_1)) = \Phi_c(p_*(\pi_1(E, e_0)))$. Therefore, we see that *as e_0 ranges over the points of $p^{-1}(b_0)$, $p_*(\pi_1(E, e_0))$ ranges precisely over a conjugacy class of subgroups of $\pi_1(B, b_0)$.*

5. HATCHER 1.3.4

[a] Let $X \subset \mathbb{R}^3$ be a union of a sphere and a diameter. Consider $\tilde{X}$ to be as in Figure 2, which is an infinite chain of spheres connected by lines. The covering p maps the spheres to original sphere and all lines to the diameter.

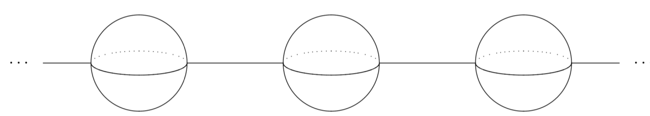

FIGURE 2. Covering for the wedge of a sphere and a diameter

$\tilde{X}$ is simply connected since it is homotopic to a wedge sum of S^2. Next we need show that p is in fact a covering map. Let $x \in X$, and let $U \ni x$ be an small open neighborhood of x. Then U can either lie completely in the sphere or the diameter, or contain the intersection of the sphere and diameter. In the both cases, $p^{-1}(U)$ are just infinite copies of U which are clearly disjoint. Each slice is clearly homeomorphic to U. Therefore, p is a covering map.

[b]

Let X now be a union of a sphere and a circle intersecting in two points. Denote the intersecting points by a, b. X is homotopic to a sphere with two disjoint curves with different direction between a and b. Each of the curve is homotopic a diameter between a and b. Therefore we can construct a covering $\tilde{X}$ as in part (a) with two direction of chains at each intersection as in Figure 3. In fact, $\tilde{X}$ can be made from replacing all nodes by spheres in the cayley graph of two generators. $\tilde{X}$ is clearly

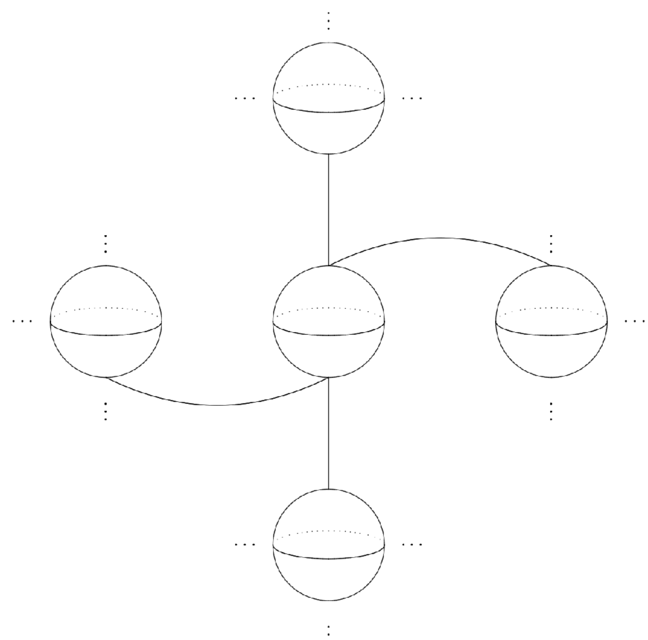

FIGURE 3. Covering for the wedge of a sphere and a circle

simply connected since it's again a wedge sum of S^2. Next, we need to show that the natural map p is a covering map. As similar argument as before, for any small open neighborhood U of $x \in X$ can either be a piece of sphere or circle, or contain the intersection of the sphere and the circle. In each case, $p^{-1}(U)$ are just countably infinite copies of U which are disjoint by construction, and each slice is clearly homeomorphic to U. Therefore p is a covering map.

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