MATH 601 ALGEBRAIC TOPOLOGY HW 6
SELECTED SOLUTIONS SKETCH/HINT

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1. Problem 9

a. Define $p : \mathbb{R} \times \mathbb{R} \to \mathcal{T}^2$ by setting $p(x, y) = (e^{2\pi ix}, e^{2\pi iy})$ where $(x, y) \in \mathbb{R}^2$. Note that $\pi_1(\mathcal{T}^2) = \mathbb{Z} \times \mathbb{Z}$. For each element $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, the correspond covering transformation in $\mathbb{R} \times \mathbb{R}$ is $h : (x, y) \to (x + m, y + n)$. It's obvious that h is a homeomorphism and satisfies $ph = p$. □

b. First the Klein bottle $\mathbf{K}$ equals $[0, 1]^2$ by identifying corresponding edges (see HW 4). Define $p : \mathbb{R} \times \mathbb{R} \to \mathbf{K}$ by the quotient map $\mathbb{R}^2 \mapsto \mathbb{R}^2/(x, y) \sim (x + 1, 1 - y)$ as shown in Figure 1. From previous HW, we know $\pi_1(\mathbf{K}) =< a, b|aba^{-1}b = 1 >$. For each element $a, b \in \pi_1(\mathbf{K})$, the

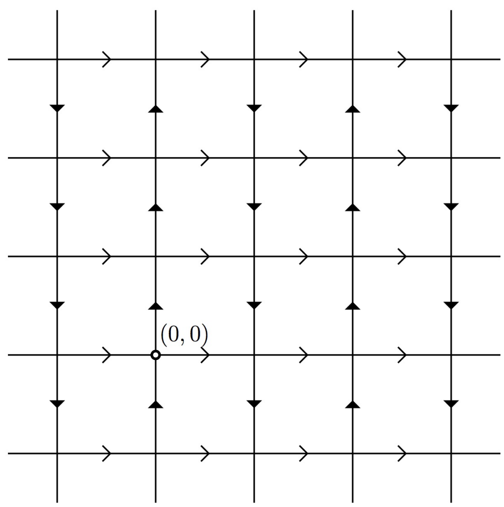

Figure 1. Universal covering space of Klein bottle $\mathbf{K}$

corresponding covering transformation is $h_a : (x, y) \mapsto (x + 1, 1 - y)$ and $h_b : (x, y) \mapsto (x, y + 1)$. □

2. Hatcher 1.3.5

Proof. Let $p : \tilde{X} \to X$ be a covering space. For every point x in the left edge $\{0\} \times I$ of X, there is a evenly covered neighborhood U_x. $\{U_x\}_x$ form an open cover for $\{0\} \times I$. By compactness, we can take a finite subcover $\{U_i\}_{i=1}^n$, then we can find a $r > 0$ such that $Y = \{(x, y) : d\big((x, y), \{0\}\times I\big) < r\}$ is contained in $\cup_{i=1}^n U_i$. We will show $Y \cap X$ is homeomorphic to its lift in $\tilde{X}$. First pick a base point $(x_0, y_0) \in p^{-1}\big((0, 0)\big)$. By unique path lifting property, we can uniquely lift $\{0\} \times I$ and $[0, r]$ into $\tilde{X}$. Let (x', y') be the end point of the lift of $\{0\}\times I$, then taking (x', y') as starting point we can uniquely lift $[0, r]\times\{1\}$. Similarly, we starting from $(1/n, 0)$ where $1/n < r$, we can uniquely lift $X\cap\{x = 1/n\}$

. It remains to show that for each $X \cap \{x = 1/n\}$, whether the lift of $(1/n, 1)$ on $X \cap \{x = 1/n\}$ lies on the lift of $[0, r] \times \{1\}$. Note that on $\Big(\cup_{i=1}^{n-1} U_i \cap (\{0\} \times I)\Big) \cup [0, 1/n] \cup \Big(\cup_{i=1}^{n-1} U_i \cap (\{1/n\} \times I)\Big)$, we have unique lift of both paths. So what we need to show is that their lift intersects in the lift of $U_n \cap X$. This is easy since p is a covering map and U_n is an evenly covered neighborhood, then $U_n \cap X$ is homeomorphic to its lift as we can specify some base point (x'', y'') in the lift of U_n where $p((x'', y'')) \in U_n \cap U_{n-1} \neq \emptyset$. Therefore the end point of the lift of $X \cap \{x = 1/n\}$ coincides with the lift of $[0, r] \times \{1\}$, and this is true for all n with $1/n \leq r$. Since on each U_i, the lift of X and X is homeomorphic, and two adjacent lifting neighborhoods agree on their intersection, the restriction of p on this lift must be 1-1. Suppose not, say $p(x) = p(y) \in X \cap Y$ and x, y lie in the lift constructed before. If $x \neq y$, there are 2 cases. If x and y lie in the same lift of U_i for some i, then since p is a local homeomorphism, $x \neq y$, which gives a contradiction. Next suppose x lies in the lift of U_i and y lies in the lift of U_j for $i \neq j$ and they do not both lie in some U_k. Since p is locally homeomorphic on U_i and U_j respectively, $U_i \cap U_j = \emptyset$, otherwise their lift will coincide in the intersection. Hence p is injective from the lift of $X \cap Y$ to $X \cap Y$. Since p is also locally homeomorphic, it's a homeomorphism.

Clearly X is not simply connected since $X \cap Y$ contains nontrivial loops. In fact, X is homotopic to an infinite wedge sum of S^1's. Since $X \cap Y$ is homeomorphic to its lift (pick arbitrary one), the lift of any nontrivial loops is $X \cap Y$ again nontrivial. Hence $\tilde{X}$ cannot be simply connected. □

3. Hatcher 1.3.9

Let $p : \mathbb{R} \to S^1$ be the covering space. Since $\pi_1(X)$ is finite and $\pi_1(S^1) = \mathbb{Z}$. But there is no group homomorphism mapping from a finite group to $\mathbb{Z}$. Hence f is zero.

So $0 = f_*(\pi_1(X)) \subset p_X(\pi_1(\mathbb{R}))$. By Lifting Criterion,$f$ lifts to $\hat{f} : X \to \mathbb{R}$. Since $\mathbb{R}$ is contractible. By results in previous chapter, $\hat{f}$ is nullhomotopic. Therefore, X is contractible, hence f is nullhomotopic.

4. Hatcher 1.3.15

By assumption $A \subset X$ and $\pi_1(\tilde{X}) = 0$. Since $p : \tilde{A} \to A$ is a covering space. Let $x_0 \in A$ be a basepoint and $\tilde{x}_0 \in \tilde{A}$ be its lift. Theorem 1.38 says that the path-connected covering spaces corresponds to a subgroup $p_*(\pi_1(\tilde{A}, \tilde{x}_0))$ of $\pi_1(A, x_0)$. Prove that the covering space $p : \tilde{A} \to A$ corresponds to the subgroup which is the kernel of $i : \pi_1(A)\pi_1(X)$, i.e. want to prove that $p_*(\pi_1(\tilde{A}, \tilde{x}_0)) = \ker i_*$.

For a loop $[\gamma]$ in $p_*(\pi_1(\tilde{A}, \tilde{x}_0))$, $[\gamma]$ lifts to a loop γ starting at $\tilde{x}_0$. Since $\tilde{A} \subset \tilde{X}$, γ is a loop in X.

But $\tilde{X}$ is simply-connected, so $\pi_1(\tilde{X}) = 0$. Hence $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = 0$ is a trivial subgroup of $\pi_1(X)$. i.e. $\tilde{X}$ corresponds to trivial subgroup of $\pi_1(X, x_0)$. Hence γ is homotopic to a constant loop in X. i.e. $[\gamma] \simeq \mathbf{1}_{x_0}$. Therefore, $[\gamma] \in \ker i_* : \pi_1(A) \to \pi_1(X)$.

The other direction is easy.

References

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