A priori Estimates

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In previous lectures we considered the maximum principle for homogeneous equations. We will now consider supremum estimates in the case of inhomogeneous equations.

**Theorem 1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Suppose \( u \in C^0(\overline{\Omega}) \cap C^2(\Omega) \) satisfies
\[
Lu = a^{ij}D_{ij}u + b^i D_i u + cu \geq f \quad \text{in } \Omega
\]
for some functions \( a^{ij}, b^i, c, \) and \( f \) on \( \Omega \). Suppose \( L \) is an elliptic operator,
\[
\beta = \sup_{\Omega} |b^i| \lambda < \infty,
\]
and \( c \leq 0 \) in \( \Omega \). Then
\[
\sup_{\Omega} u \leq \sup_{\partial \Omega} u^+ + C \sup_{\Omega} \left| \frac{f^-}{\lambda} \right|
\]
for \( C = e^{(\beta+1)d} - 1 \), where \( d = \text{diam} \Omega \) and \( f^- = \max\{-f, 0\} \). Note that if \( Lu = f \) in \( \Omega \), we have
\[
\sup_{\Omega} |u| \leq \sup_{\partial \Omega} |u| + C \sup_{\Omega} \left| \frac{f}{\lambda} \right|
\]
for \( C = e^{(\beta+1)d} - 1 \).

**Proof.** Without loss of generality let \( \Omega \) lie between the slab \( 0 < x_1 < d \). Set \( L_0 = a^{ij}D_{ij} + b^i D_i \). Let
\[
v = \sup_{\partial \Omega} u^+ + (e^{\alpha d} - e^{\alpha x_1}) \sup_{\partial \Omega} \frac{|f^-|}{\lambda},
\]
where \( \alpha \geq \beta + 1 \). We claim that \( Lu \geq f \geq Lv \) in \( \Omega \), in which case we can apply the comparison principle using the fact that \( u \leq v \) on \( \partial \Omega \) to conclude that \( u \leq v \) on \( \overline{\Omega} \). For \( \alpha \geq \beta + 1 \) we have
\[
L_0 e^{\alpha x_1} = (\alpha^2 a^{11} + \alpha b^1) e^{\alpha x_1} \geq \lambda (\alpha^2 - \alpha \beta) e^{\alpha x_1} \geq \lambda.
\]
Thus
\[
Lv = L_0 v + cv \leq L_0 v \quad \text{(since } c \leq 0, v \geq 0 \text{ in } \Omega) \]
\[
= -L_0(e^{\alpha x_1}) \sup_{\partial \Omega} \frac{|f^-|}{\lambda} \quad \text{(by linearity of } L) \]
\[
\leq -\lambda \sup_{\partial \Omega} \frac{|f^-|}{\lambda} \quad \text{(since } L_0(e^{\alpha x_1}) \geq \lambda \text{ in } \Omega) \]
\[
\leq f
\]
in $\Omega$. By the comparison principle, $u \leq v$ on $\Omega$. In particular,

$$\sup_{\Omega} u \leq \sup_{\partial \Omega} v = \sup_{\Omega} u^+ + (e^{\alpha d} - 1) \sup_{\Omega} \frac{|f^-|}{\lambda}.$$ 

Replacing $u$ with $-u$ completes the proof in the case that $Lu = f$ in $\Omega$. 

**Corollary 1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Suppose $u \in C^0(\Omega) \cap C^2(\Omega)$ satisfies

$$Lu = a^{ij}D_{ij}u + b^iD_iu + cu = f \text{ in } \Omega$$

for some functions $a^{ij}$, $b^i$, $c$, and $f$ on $\Omega$. Suppose $L$ is an elliptic operator and

$$\beta = \sup_{\Omega} \frac{|b^i|}{\lambda} < \infty.$$

Suppose that $\Omega$ is a small enough domain that

$$\gamma = (e^{(\beta+1)d} - 1)\frac{c_+}{\lambda} < 1,$$

where $d = \text{diam } \Omega$ and $c = c_+ - c_-$ for $c_+ = \max\{c, 0\}$ and $c_- = \max\{-c, 0\}$. Then

$$\sup_{\Omega} u \leq \frac{1}{1 - \gamma} \left( \sup_{\partial \Omega} u^+ + C \sup_{\Omega} \frac{|f^-|}{\lambda} \right)$$

for some constant $C \in (0, \infty)$ depending only on $\beta$ and $d$.

**Proof.** Observe that

$$a^{ij}D_{ij}u + b^iD_iu - c_- u \geq f - c_+ u \text{ in } \Omega.$$

By Theorem 1,

$$\sup_{\Omega} |u| \leq \sup_{\partial \Omega} |u| + C \sup_{\Omega} \frac{|f|}{\lambda} + C \sup_{\Omega} \frac{c_+}{\lambda} \sup_{\Omega} |u|$$

$$\leq \sup_{\partial \Omega} |u| + C \sup_{\Omega} \frac{|f|}{\lambda} + \gamma \sup_{\Omega} |u|.$$ 

Since $\gamma < 1$,

$$\sup_{\Omega} |u| \leq \frac{1}{1 - \gamma} \left( \sup_{\partial \Omega} u^+ + C \sup_{\Omega} \frac{|f^-|}{\lambda} \right).$$

**Corollary 2.** (Uniqueness of Solutions to the Dirichlet Problem on Small Domains) Let $\Omega$ be a bounded open set in $\mathbb{R}^n$. Consider the Dirichlet problem

$$Lu = a^{ij}D_{ij}u + b^iD_iu + cu = f \text{ in } \Omega,$$

$$u = \varphi \text{ on } \partial \Omega,$$

for some functions $a^{ij}$, $b^i$, $c$, and $f$ on $\Omega$ and $\varphi \in C^0(\partial \Omega)$ such that $L$ is an elliptic operator and

$$\beta = \sup_{\Omega} \frac{|b^i|}{\lambda} < \infty, \quad \sup_{\Omega} \frac{|c|}{\lambda} < \infty.$$
Suppose that $\Omega$ is a small enough domain that

$$\gamma = (e^{(\beta+1)d} - 1)\frac{c_+}{\lambda} < 1,$$

where $d = \text{diam } \Omega$ and $c = c_+ - c_-\text{ for } c_+ = \max\{c, 0\}$ and $c_- = \max\{-c, 0\}$. Then there is at most one solution $u \in C^0(\Omega) \cap C^2(\Omega)$ to the Dirichlet problem (i.e. there may be no solution or a unique solution but there cannot be two or more solutions).

**Proof.** Suppose $u_1$ and $u_2$ are two solutions to the Dirichlet problem. Then

\begin{align*}
L(u_1 - u_2) &= 0 \text{ in } \Omega, \\
u_1 - u_2 &= 0 \text{ on } \partial \Omega.
\end{align*}

By Corollary 1, $u_1 - u_2 = 0$ on $\overline{\Omega}$, i.e. $u_1 = u_2$ on $\overline{\Omega}$. \hfill \qed

**References:** Gilbarg and Trudinger, Section 3.3.