1 Definitions and notation

Let $\Omega$ be a domain in $\mathbb{R}^n$. We say that $u \in W^{1,2}(\Omega)$ satisfies

$$Lu = D_i(a^{ij}D_ju + b^iu) + c^jD_ju + du = D_i f^i + g \text{ weakly in } \Omega,$$

where the coefficients $a^{ij}$, $b^i$, $c^j$, and $d$ are measurable functions on $\Omega$ and $f^i, g \in L^2(\Omega)$, if

$$\int_{\Omega} ((a^{ij}D_ju + b^iu)D_i\zeta - (c^jD_ju + du)\zeta) = \int_{\Omega} (f^iD_i\zeta - g\zeta)$$

for all test functions $\zeta \in C^\infty_c(\Omega)$. We call $u$ a weak solution to (1). Note that if (2) holds true for all $\zeta \in C^\infty_c(\Omega)$, then by a continuity argument using $C^\infty_c(\Omega)$ being dense in $W^{1,2}_0(\Omega)$, (2) holds true for all $\zeta \in W^{1,2}_0(\Omega)$.

Observe that if the functions $u, a^{ij}, b^i, c^j, d, f^i, g$ were sufficiently smooth on $\Omega$, for example $u \in C^2(\Omega)$, $a^{ij}, b^i, f \in C^1(\Omega)$, and $c^j, d, g \in C^0(\Omega)$, then by integration by parts,

$$Lu = D_i(a^{ij}D_ju + b^iu) + c^jD_ju + du = D_i f^i + g \text{ pointwise in } \Omega$$

implies that (2) holds true and conversely (2) implies that

$$-\int_{\Omega} Lu\zeta = -\int_{\Omega} (D_i f^i + g)\zeta$$

for all $\zeta \in C^\infty_c(\Omega)$, which since $\zeta$ is arbitrary implies (3). However, (3) does not make sense under the weaker regularity conditions that $u \in W^{1,2}(\Omega)$, $a^{ij}, b^i, c^j, d$ are measurable functions on $\Omega$, and $f^i, g \in L^2(\Omega)$, whereas (2) does make sense under the weaker regularity conditions.

We shall assume the ellipticity condition

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^n$$

for some constant $\lambda > 0$. Note that for equations in divergence form we cannot assume that $a^{ij}(x) = a^{ji}(x)$ for a.e. $x \in \Omega$. It will be standard to assume that the coefficients are bounded with

$$\sum_{i,j=1}^n |a^{ij}(x)|^2 \leq \Lambda^2, \quad \lambda^{-2} \sum_{i=1}^n (|b^i(x)|^2 + |c^i(x)|^2) + \lambda^{-1}|d^i(x)| \leq \nu^2 \text{ for a.e. } x \in \Omega$$
for some constants $\Lambda, \nu \in (0, \infty)$.

We can similarly consider differential inequalities

$$Lu = D_i(a^{ij}D_j u + b^i u) + c^j D_j u + du \geq (\leq)D_i f^i + g \text{ weakly in } \Omega$$

for $u \in W^{1,2}(\Omega)$, which we take to mean that

$$\int_\Omega ((a^{ij}D_j u + b^i u)D_i \zeta - (c^j D_j u + du)\zeta) \leq (\geq) \int_\Omega (f^i D_i \zeta - g\zeta)$$

(6)

for all non-negative $\zeta \in C^\infty_c(\Omega)$ (or equivalently for all $\zeta \in W^{1,2}_0(\Omega)$).

We also want to consider the Dirichlet problem

$$D_i(a^{ij}D_j u + b^i u) + c^j D_j u + du = D_i f^i + g \text{ weakly in } \Omega,$$

$$u = \varphi \text{ on } \partial \Omega,$$

where $u \in W^{1,2}(\Omega)$, the coefficients $a^{ij}, b^i, c^j,$ and $d$ are bounded measurable functions on $\Omega$, $f^i, g \in L^2_{loc}(\Omega)$, and $\varphi \in W^{1,2}(\Omega)$. By $u = \varphi$ on $\partial \Omega$, we mean that

$$u - \varphi \in W^{1,2}_0(\Omega).$$

Note that if $\Omega, u,$ and $\varphi$ are sufficiently smooth, namely $\Omega$ is a $C^1$ domain and $u, \varphi \in C^1(\overline{\Omega})$, then $u - \varphi \in W^{1,2}_0(\Omega)$ implies that $u = \varphi$ pointwise on $\partial \Omega$. To see this, recall that $u - \varphi \in W^{1,2}_0(\Omega)$ means that there exists a sequence of functions $v_j \in C^\infty_c(\Omega)$ such that $v_j \to u - \varphi$ in $W^{1,2}(\Omega)$. Thus

$$\int_{\partial \Omega} (u - \varphi) \zeta \cdot \nu = \int_\Omega (D(u - \varphi) \cdot \zeta + (u - \varphi) \text{ div } \zeta)$$

$$= \lim_{j \to \infty} \int_\Omega (Dv_j \cdot \zeta + v_j \text{ div } \zeta)$$

$$= \lim_{j \to \infty} \int_{\partial \Omega} v_j \zeta \cdot \nu$$

$$= 0,$$

for all $\zeta \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n)$, where $\nu$ denotes the outward unit normal to $\partial \Omega$. Since $\zeta$ is arbitrary, $u = \varphi$ pointwise on $\partial \Omega$.

2 Maximum principle

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Let $u \in W^{1,2}(\Omega)$. By $\sup_{\Omega}$ $u$ we mean the essential supremum, i.e.

$$\sup_{\Omega} u = \inf\{k \in \mathbb{R} : u \leq k \text{ a.e. in } \Omega\}.$$

By $\sup_{\partial \Omega}$ $u$, we mean

$$\sup_{\partial \Omega} u = \inf\{k \in \mathbb{R} : (u - k)^+ \in W^{1,2}_0(\Omega)\},$$

where $v^+(x) = \max\{v(x), 0\}$ for measurable functions $v$ on $\Omega$. Using the fact that $\lim_{\varepsilon \to 0} L^n(\{x \in \Omega : 0 < u(x) - \sup_{\partial \Omega} u < \varepsilon\}) = 0$ and $W^{1,2}_0(\Omega)$ is closed in $W^{1,2}(\Omega)$, given easy to see that $(u - k)^+$
converges to $(u - \sup_{\partial \Omega} u)^+$ in $W^{1,2}_0(\Omega)$ as $k \downarrow \sup_{\partial \Omega} u$, so in particular $(u - \sup_{\partial \Omega} u)^+ \in W^{1,2}_0(\Omega)$.

We can similarly define
\[
\inf_{\Omega} u = \sup\{k \in \mathbb{R} : u \geq k \text{ a.e. in } \Omega\},
\]
\[
\inf_{\partial \Omega} u = \sup\{k \in \mathbb{R} : (u - k)^- \in W^{1,2}_0(\Omega)\}.
\]

where $v^-(x) = \min\{v(x), 0\}$ for measurable functions $v$ on $\Omega$. Given $u \in W^{1,2}(\Omega)$, obviously $u \leq v$ means that $u \leq v$ a.e. in $\Omega$. We say $u \leq v$ on $\partial \Omega$ if $(u - v)^+ \in W^{1,2}_0(\Omega)$.

**Theorem 1** (Weak maximum principle). Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Suppose $u \in W^{1,2}(\Omega)$ satisfies
\[
Lu = D_i(a^{ij}D_j u + b^i u) + c^i D_j u + du \geq 0 \text{ in } \Omega
\]
where $a^{ij}$, $b^i$, $c^i$, and $d$ are measurable function on $\Omega$ satisfying (4) and (5) for some constants $0 < \lambda, \Lambda, \nu < \infty$ and
\[
\int_{\Omega} (-b^i D_i \zeta + d \zeta) \leq 0
\]
for all nonnegative $\zeta \in W^{1,1}_0(\Omega)$. Then
\[
\sup_{\partial \Omega} u \leq \sup_{\partial \Omega} u^+,
\]
where $u^+(x) = \max\{u(x), 0\}$ for $x \in \Omega$.

Heuristically,
\[
Lu = D_i(a^{ij}D_j u + b^i u) + c^i D_j u + du = a^{ij}D_{ij}u + (D_i a^i + b^i + c^i)D_i u + (D_i b^i + d)u \text{ in } \Omega.
\]

Since if $b^i$ and $d$ are sufficiently smooth ($b^i \in W^{1,1}(\Omega)$ and $d \in L^1(\Omega)$ is sufficient), then by integration by parts $D_i b^i + d \leq 0$ a.e. in $\Omega$ is equivalent to (7). Thus we can interpret (7) as meaning that $D_i b^i + d \leq 0$ weakly in $\Omega$. (7) is the analogue to $c \leq 0$ in the case of the classical elliptic operator $Lu = a^{ij}D_{ij}u + b^i D_i u + cu$.

**Proof of the weak maximum principle.** We will use a standard type of proof technique using the weak inequality
\[
\int_{\Omega} ((a^{ij}D_j u + b^i u)D_i \zeta - (c^i D_j u + du) \zeta) \leq 0
\]
for all nonnegative $\zeta \in W^{1,2}_0(\Omega)$.

Our first step is to use (7) to simplify the inequality. By rewriting (8) and using (7),
\[
\int_{\Omega} (a^{ij}D_j uD_i \zeta - (b^i + c^i)D_j u \zeta) \leq \int_{\Omega} (-b^i D_i (u \zeta) + d(u \zeta)) \leq 0.
\]
for all $\zeta \in W^{1,2}_0(\Omega)$ such that $\zeta \geq 0$ and $u \zeta \geq 0$ a.e. in $\Omega$. Note that $u \in W^{1,2}(\Omega)$ and $\zeta \in W^{1,2}_0(\Omega)$ implies that $u \zeta \in W^{1,1}_0(\Omega)$.

The case where $b^i + c^i = 0$ a.e. in $\Omega$ is particularly easy. We now will chose a particular test function $\zeta$ in (9), namely $\zeta = (u - l)^+$ for $l = \sup_{\partial \Omega} u^+$. Note that this $\zeta$ is indeed in $W^{1,2}_0(\Omega)$. By (9) obtain
\[
\int_{\Omega} a^{ij}D_j \zeta D_i \zeta \leq 0.
\]
By (4),
\[ \lambda \int_{\Omega} |D\zeta|^2 \leq 0, \]
so \( D\zeta = 0 \) a.e. in \( \Omega \). Thus \( \zeta \) is constant on \( \Omega \). In particular, since \( \zeta \in W^{1,2}_0(\Omega) \), \( \zeta = 0 \) a.e. in \( \Omega \).

Therefore
\[ \sup_{\Omega} u \leq l = \sup_{\partial \Omega} u^+. \]

Now suppose \( b^i + c^i \) is not identically zero on \( \Omega \). By way of contradiction suppose that
\[ \sup_{\partial \Omega} u^+ < \sup_{\Omega} u. \]

Let \( l \in \mathbb{R} \) such that
\[ \sup_{\partial \Omega} u^+ < l < \sup_{\Omega} u. \]

Now we proceed with a standard type of argument. Like before, we choose our test function \( \zeta \), in particular we choose \( \zeta = (u - l)^+ \) in (9). We note that \( \zeta \in W^{1,2}_0(\Omega) \). Then by (9)
\[ \int_{\Omega} a^{ij} D_j \zeta D_i \zeta - (b^i + c^i) \zeta D_j \zeta \leq 0. \]

Next we rewrite this inequality as
\[ \int_{\Omega} a^{ij} D_j \zeta D_i \zeta \leq \int_{\Omega} (b^i + c^i) \zeta D_j \zeta \]
so that the integral of \( a^{ij} D_j \zeta D_i \zeta \) is on the left hand side and everything else is on the right hand side. Then by (4) and (5),
\[ \lambda \int_{\Omega} |D\zeta|^2 \leq 2\lambda\nu \int_{\Omega} |D\zeta|. \]

Next we move all the \( D\zeta \) terms to the left hand side using the Cauchy inequality \( ab \leq \frac{1}{4}a^2 + b^2 \) for \( a, b \geq 0 \) to get
\[ \lambda \int_{\Omega} |D\zeta|^2 \leq \frac{\lambda}{2} \int_{\Omega} |D\zeta|^2 + 2\lambda\nu^2 \int_{\Gamma} |\zeta|^2 \]
where \( \Gamma = \{ x \in \Omega : D\zeta(x) \neq 0 \} \), and then move the integral of \( |D\zeta|^2 \) to the left hand side to get
\[ \int_{\Omega} |D\zeta|^2 \leq 4\nu^2 \int_{\Gamma} |\zeta|^2. \quad (10) \]

Note that here we used the fact that \( |D\zeta| = 0 \) on \( \Omega \setminus \Gamma \) to get an integral over \( \Gamma \) on the right hand side of (10). This will be important in a moment. The next step is to apply the Sobolev inequality on the left hand side to obtain
\[ \frac{1}{C^2} \|\zeta\|_{L^{2(n/(n-2))}}^2 \leq 4\nu^2 \int_{\Gamma} |\zeta|^2 \]
for some constant \( C = C(n) \in (0, \infty) \) and then apply the Hölder inequality to the right hand side to obtain
\[ \frac{1}{C^2} \|\zeta\|_{L^{2(n/(n-2))}}^2 \leq 4\nu^2 |\Gamma|^{2/n} \|\zeta\|_{L^{2(n/(n-2))}}^2, \]
where $|S|$ denotes the Lebesgue measure of a set $S$, which by cancelling $\|\zeta\|_{L^{2n/(n-2)}} > 0$ implies

$$ (2C\nu)^{-n} \leq |\Gamma|. $$

(11)

Now the application of the Sobolev inequality to the left hand side of (10) only makes sense if $n > 2$. If $n = 2$, let $1 < \hat{n} < 2$, and note that by the (10), Sobolev inequality, and the Hölder inequality

$$ \frac{1}{C} \|\zeta\|_{L^{2n/(2-\hat{n})}(\Gamma)} \leq \|Du\|_{L^2(\Gamma)} $$

$$ \leq |\Gamma|^{1/\hat{n}-1/2} \|D\zeta\|_{L^2(\Gamma)} $$

$$ \leq 2\nu|\Gamma|^{1/\hat{n}-1/2} \|\zeta\|_{L^2(\Gamma)} \text{ (by (10))} $$

so cancelling $\|\zeta\|_{L^{2n/(2-\hat{n})}(\Gamma)}$ yields (11) in the case $n = 2$. Since $\Gamma$ is where $\zeta = (u-l)^+$ satisfies $\zeta \geq 0$ and $Du \neq 0$, we can rewrite (11) as

$$ C \leq \{|x : u(x) \geq l, Du(x) \neq 0\}. $$

(12)

Note that the set on the right hand side of (12) is decreasing (with respect to set inclusion $\subseteq$) as $l$ increases. If $\sup_{\Omega} u = \infty$, let $l \uparrow \infty$ in (12) to obtain $u(x) = \infty$ on a subset of $\Omega$ with positive measure, which contradicts $u \in L^2(\Omega)$. If $\sup_{\Omega} u < \infty$, let $l$ increase to $\sup_{\Omega} u$ in (12), we obtain $u = \sup_{\Omega} u$ and $Du \neq 0$ on a subset of $\Omega$ of positive measure, which is impossible by Lemma 1 below.

**Lemma 1.** Let $u \in W^{1,2}(\Omega)$. If $u$ is constant on some measurable set $S$ in $\Omega$, then $Du = 0$ a.e. on $S$.

**Proof.** WLOG suppose $u = 0$ on $S$.

Recall that if $f : \mathbb{R} \to \mathbb{R}$ is a $C^1$ function with bounded derivative and $u \in W^{1,2}(\Omega)$, then $f(u) \in W^{1,2}(\Omega)$ with weak derivative $f'(u)Du$. We claim that for $u^+(x) = \max\{u(x), 0\}$, $u^+ \in W^{1,2}(\Omega)$ with $Du^+(x) = Du(x)$ at a.e. $x \in \Omega$ with $u(x) > 0$ and $Du^+(x) = 0$ for a.e. $x \in \Omega$ with $u(x) \leq 0$. To see this, for $\varepsilon > 0$ let $f_{\varepsilon} : \mathbb{R} \to \mathbb{R}$ a smooth, convex function with $f_{\varepsilon}(t) = 0$ for $t \leq \varepsilon/2$ and $f_{\varepsilon}(t) = 1$ for $t \geq \varepsilon$. Fix a test function $\zeta \in C_\infty(\Omega)$. Since $f_{\varepsilon}(t) = 0$ for $t \leq 0$ and $t - \varepsilon < f_{\varepsilon}(t) \leq t$ for $t \geq 0$,

$$ \left| \int_{\Omega} u^+D\zeta - \int_{\Omega} f_{\varepsilon}(u)Du\zeta \right| \leq \varepsilon \int_{\Omega} |\zeta| \to 0 $$

as $\varepsilon \downarrow 0$. Hence

$$ -\int_{\Omega} u^+D\zeta = \lim_{\varepsilon \downarrow 0} -\int_{\Omega} f_{\varepsilon}(u)Du\zeta = \lim_{\varepsilon \downarrow 0} \int_{\Omega} f_{\varepsilon}'(u)Du\zeta = \int_{\Omega \cap \{u > 0\}} Du\zeta, $$

where the last step follows from the dominated convergence theorem and the fact that $f_{\varepsilon}'(u) = 0$ if $u \leq 0$ and $f_{\varepsilon}'(u) \uparrow 1$ if $u > 0$.

By the same argument, we can show that for $u^-(x) = \min\{u(x), 0\}$, $u^- \in W^{1,2}(\Omega)$ with $Du^-(x) = Du(x)$ at a.e. $x \in \Omega$ with $u(x) < 0$ and $Du^-(x) = 0$ for a.e. $x \in \Omega$ with $u(x) \geq 0$.

Since $u = 0$ a.e. on $S$, $Du^+ = Du^- = 0$ a.e. on $S$. Since $u = u^+ + u^-$, $Du = Du^+ + Du^- = 0$ a.e. on $S$.

**Corollary 1** (Uniqueness for the Dirichlet Problem). Consider $L$ as above (i.e., satisfying (4), (5), and (7).) Suppose $u, v \in W^{1,2}(\Omega)$ such that $Lu = Lv$ in $\Omega$ and $u = v$ on $\partial\Omega$ (i.e. $u - v \in W^{1,2}_0(\Omega)$). Then $u = v$ a.e. in $\Omega$. 


3 Existence Theory

Recall the following:

**Theorem 2** (Lax Milgram). Let \( \mathcal{H} \) be a Hilbert space, \( B : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \) be a bilinear functional that is

**Bounded:** \( |B(x, y)| \leq C_1 \|x\|_\mathcal{H} \|y\|_\mathcal{H} \) for all \( x, y \in \mathcal{H} \) for some constant \( C_1 \in (0, \infty) \) and

**Coercive:** \( B(x, x) \geq C_2 \|x\|_\mathcal{H}^2 \) for all \( x \in \mathcal{H} \) for some constant \( C_2 \in (0, \infty) \).

Let \( F : \mathcal{H} \to \mathbb{R} \) be a bounded linear functional on \( \mathcal{H} \). Then there exists a unique element \( z \in \mathcal{H} \) such that

\[
B(z, x) = F(x) \quad \text{for all } x \in \mathcal{H}.
\] (13)

Moreover, \( \|z\|_\mathcal{H} \leq (1/C_2) \|F\| \). (References: Gilbarg and Trudinger, Theorem 5.8)

**Proof.** For every \( x \in \mathcal{H} \), by Riesz representation applied to the bounded linear functional \( B(x, .) \), there is a unique element \( Tx \in \mathcal{H} \) such that \( B(x, y) = (Tx, y)_\mathcal{H} \) and \( \|B(x, .)\|_{\mathcal{H}^*} = \|Tx\|_\mathcal{H} \). Since \( B \) is bilinear, \( T : \mathcal{H} \to \mathcal{H} \) is linear. Also by Riesz representation applied to \( F \), there is a unique \( w \in \mathcal{H} \) such that \( F(x) = (w, x) \) for all \( x \in \mathcal{H} \). Thus (13) is equivalent to

\[
(Tz, x) = (w, x) \quad \text{for all } x \in \mathcal{H},
\] (14)

which is in turn equivalent to

\[
Tz = w.
\] (15)

((14) implies (15) by choosing \( x = Tz - w \).) Thus in order to show that there is a solution \( z \) to (13) with \( \|z\|_\mathcal{H} \leq (1/C_2) \|F\| \), it suffices to show that \( T \) has an inverse function \( T^{-1} : \mathcal{H} \to \mathcal{H} \) which is a bounded linear map with norm \( \|T^{-1}\| \leq 1/C_2 \).

Since \( B \) is coercive,

\[
C_2 \|x\|_{\mathcal{H}}^2 \leq B(x, x) = (x, Tx) \leq \|x\|_\mathcal{H} \|Tx\|_\mathcal{H},
\]

so

\[
C_2 \|x\|_\mathcal{H} \leq \|Tx\|_\mathcal{H}.
\] (16)

Now we use (16) to show that \( T : \mathcal{H} \to \mathcal{H} \) is bijective and \( \|T^{-1}\| \leq 1/2C_2 \):

(1) \( T \) is injective: If \( Tx_1 = Tx_2 \) for some \( x_1, x_2 \in \mathcal{H} \) then \( T(x_1 - x_2) = 0 \) in \( \mathcal{H} \) and by (16) \( \|x_1 - x_2\|_\mathcal{H} = 0 \), so \( x_1 = x_2 \).

(2) \( T \) has closed range: Suppose \( x_j \in \mathcal{H} \) such that \( Tx_j \to y \) in \( \mathcal{H} \). Then by (16)

\[
\|x_j - x_k\|_\mathcal{H} \leq \frac{1}{C_2} \|Tx_j - Tx_k\|_\mathcal{H} \to 0
\]

as \( j, k \to \infty \), so \( x_j \) is Cauchy. Thus \( x_j \) converges to some \( x \) in \( \mathcal{H} \) and \( y = Tx \).

(3) The range \( T(\mathcal{H}) \) of \( T \) is \( \mathcal{H} \): Suppose there is a \( x \in T(\mathcal{H})^\perp \). By coercivity,

\[
C_2 \|x\|_{\mathcal{H}}^2 \leq B(x, x) = (Tx, x)_\mathcal{H} = 0,
\]

so \( x = 0 \). Therefore \( T(\mathcal{H}) = \mathcal{H} \).
(4) $\|T^{-1}\| \leq 1/C_2$: Let $y = T^{-1}x$. Then by (16),

$$\|T^{-1}x\|_H = \|y\|_H \leq \frac{1}{C_2} \|Ty\|_H = \frac{1}{C_2} \|x\|_H.$$  

\[\square\]

**Theorem 3.** Let $\Omega$ be a domain in $\mathbb{R}^n$. Let $a^{ij}, b^i, c^j, d \in L^\infty(\Omega)$ be coefficients satisfying (4) and (5) for some constants $0 < \lambda, \Lambda, \nu < \infty$ and (7). For every $f^i, g \in L^2(\Omega)$ and $\varphi \in W^{1,2}(\Omega)$, there is a unique solution $u \in W^{1,2}(\Omega)$ to the Dirichlet problem

$$Lu = D_i f^i + g \text{ weakly in } \Omega,$$

$$u = \varphi \text{ on } \partial \Omega.$$  

Moreover,

$$\|u\|_{W^{1,2}(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} + \|\varphi\|_{W^{1,2}(\Omega)}).$$

**Proof.** By the maximum principle, the solution to the Dirichlet problem is unique if it exists, so what remains to show is the existence of solutions. By replacing $u$ with $v = u - \varphi$ and solving for $v$ such that $Lv = D_i f^i + g - L\varphi$ weakly in $\Omega$ and $v = 0$ on $\partial \Omega$, it suffices to assume that $\varphi = 0$ a.e. on $\Omega$.

Define the bounded bilinear functional $L : W^{1,2}_0(\Omega) \times W^{1,2}_0(\Omega) \to \mathbb{R}$ by

$$L(u, v) = \int_\Omega \left( a^{ij} D_i u D_j v + b^i u D_i v - c^j D_j uv - dv \right)$$

and define the bounded linear functional $F : W^{1,2}_0(\Omega) \to \mathbb{R}$ by

$$F(\zeta) = \int_\Omega (f^i D_i \zeta - g \zeta).$$

Clearly solving for $u \in W^{1,2}(\Omega)$ satisfying (17) with $\varphi = 0$ is equivalent to solving for $u \in W^{1,2}_0(\Omega)$ such that

$$L(u, \zeta) = F(\zeta) \text{ for all } \zeta \in W^{1,2}_0(\Omega).$$  

(18)

By Lax-Milgram, it suffices to show that $L$ is coercive. Unfortunately, we only have

$$L(v, v) = \int_\Omega \left( a^{ij} D_i v D_j v + (b^i - c^i) v D_i v - dv^2 \right)$$

$$\geq \lambda \int_\Omega |Dv|^2 - \lambda \int_\Omega (2\nu|v||Dv| + \nu^2|v|^2) \text{ (by (4) and (5))}$$

$$\geq \frac{\lambda}{2} \int_\Omega |Dv|^2 - 3\lambda \nu^2 \int_\Omega v^2 \text{ (by Cauchy's inequality),}$$

so $L$ is not necessarily coercive.

If we instead considered the problem of solving for $u \in W^{1,2}_0(\Omega)$ such that

$$L_\sigma u \equiv Lu - \sigma u = D_i f^i + g \text{ weakly in } \Omega.$$  

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for given $f^i, g \in W_0^{1,2}(\Omega)$, then the corresponding bilinear form

$$\mathcal{L}_\sigma(u, v) = \int_{\Omega} (a^{ij} D_j u D_i v + b^i u D_i v - c^j D_j u v + (-d + \sigma) u v)$$

would satisfy

$$\mathcal{L}_\sigma(v, v) \geq \frac{\lambda}{2} \int_{\Omega} |Dv|^2 + (\sigma - 3\lambda \nu^2) \int_{\Omega} v^2,$$

so $\mathcal{L}_\sigma$ is obviously coercive provided $\sigma$ is sufficiently large. By Lax-Milgram, there exists an inverse map $L^{-1}_\sigma : W_0^{1,2}(\Omega)^* \to W_0^{1,2}(\Omega)$ such that for every bounded linear functional $F \in W_0^{1,2}(\Omega)^*$, $u = TF$ is the solution to $L_\sigma u = F$ weakly in $\Omega$, i.e.

$$\mathcal{L}_\sigma(u, \zeta) = F(\zeta) \text{ for all } \zeta \in W_0^{1,2}(\Omega).$$

Observe that $Lu = F$ weakly in $\Omega$, where $F \in W_0^{1,2}(\Omega)^*$, is equivalent to

$$u + \sigma L^{-1}_\sigma u = L^{-1}_\sigma F \text{ in } \Omega. \quad (19)$$

We know

$$L^{-1}_\sigma : W_0^{1,2}(\Omega) \subset L^2(\Omega) \subset W_0^{1,2}(\Omega)^* \to W_0^{1,2}(\Omega),$$

which is compact since the embedding $W_0^{1,2}(\Omega) \subset L^2(\Omega)$ is compact by Rellich’s lemma. Note that here the embedding $L^2(\Omega) \subset W_0^{1,2}(\Omega)^*$ is defined by mapping $v \in L^2(\Omega)$ to the linear functional $\zeta \mapsto \int_{\Omega} v \zeta$. Since $L^{-1}_\sigma$ is a compact linear operator between Banach spaces, by spectral theory for $L^{-1}_\sigma$ either $-1/\sigma$ is an eigenvalue of $L^{-1}_\sigma$ or (19) has a unique solution $u \in W_0^{1,2}(\Omega)$ for all $F \in W_0^{1,2}(\Omega)^*$ and $\|u\|_{W^{1,2}(\Omega)} \leq C\|F\|$ for some $C = C(\lambda, \Lambda, \nu) \in (0, \infty)$. Since the solution to the Dirichlet problem for $L$ is unique by the maximum principle, in particular $Lu = 0$ weakly in $\Omega$ only when $u = 0$. $-1/\sigma$ is not an eigenvalue of $L^{-1}_\sigma$ and thus there exists a unique solution $u \in W_0^{1,2}(\Omega)$ to $Lu = F$ weakly in $\Omega$ for every $F \in W_0^{1,2}(\Omega)^*$.

The spectral theory for $L^{-1}_\sigma$, we obtain the Fredholm alternative for equations in divergence form:

**Theorem 4** (Fredholm alternative). Let

$$Lu = D_i (a^{ij} D_j u + b^i u) + c^j D_j u + du \text{ in } \Omega$$

for $u \in W^{1,2}(\Omega)$, where $a^{ij}, b^i, c^j, d \in L^\infty(\Omega)$ satisfying (4) and (5). There exists a countable, discrete set $\Sigma \subset \mathbb{R}$ such that

(a) if $\lambda \not\in \Sigma$, the Dirichlet problem, $Lu + \lambda u = D_i f^i + g$ in $\Omega$, $u = \varphi$ on $\partial \Omega$, has a unique solution $u \in W^{1,2}(\Omega)$ for all $f^i, g \in L^2(\Omega)$ and $\varphi \in W^{1,2}(\Omega)$, and

(b) if $\lambda \in \Sigma$, the homogeneous problem, $Lu + \lambda u = 0$ in $\Omega$, $u = 0$ on $\partial \Omega$, has a finite dimensional subspace of nontrivial solutions $u \in W_0^{1,2}(\Omega)$. We call $\lambda$ a Dirichlet eigenvalue of $L$.

(Note that some books, for example Gilbarg and Trudinger, define $\Sigma$ as the set of $\lambda$ such that there is a nontrivial solution $u \in W_0^{1,2}(\Omega)$ to $Lu - \lambda u = 0$ in $\Omega$.)
4 Regularity theory

Theorem 5 ($W^{2,2}$ Interior Regularity). Let $\Omega$ be an open set in $\mathbb{R}^n$. Suppose $u \in W^{1,2}(\Omega)$ satisfies
\[ Lu = D_i(a^{ij}D_ju + b^i u) + c^j D_ju + du = f \text{ weakly in } \Omega \]
for an elliptic operator $L$ with coefficients $a^{ij}, b^i \in C^{0,1}(\Omega)$ and $c^j \in L^\infty_{loc}(\Omega)$ and a function $f$ with $f \in L^2_{loc}(\Omega)$. By elliptic, we just require that $a^{ij}(x)x_i x_j \geq \lambda |\xi|^2$ for some $\lambda \in (0, \infty)$. Then $u \in W^{2,2}_{loc}(\Omega)$ with
\[ \|u\|_{W^{2,2}(\Omega')} \leq C(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}) \]
for every $\Omega' \subset \subset \Omega$ for some constant $C = C(n, L, \Omega, \Omega) \in (0, \infty)$.

Theorem 6 ($W^{2+k,2}$ Interior Regularity for $k \geq 1$). Let $k \geq 1$ be an integer. Let $\Omega$ be an open set in $\mathbb{R}^n$. Suppose $u \in W^{1,2}(\Omega)$ satisfies
\[ Lu = D_i(a^{ij}D_ju + b^i u) + c^j D_ju + du = f \text{ weakly in } \Omega \]
for an elliptic operator $L$ with coefficients $a^{ij}, b^i \in C^{k,1}(\Omega)$ and $c^j \in C^{k-1,1}(\Omega)$ and $f \in W^{k,2}_{loc}(\Omega)$. Then $u \in W^{k+2,2}_{loc}(\Omega)$ with
\[ \|u\|_{W^{k+2,2}(\Omega')} \leq C(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{W^{k,2}(\Omega)}) \]
if $k \geq 1$ for every $\Omega' \subset \subset \Omega$ for some constant $C = C(n, k, L, \Omega, \Omega) \in (0, \infty)$.

Moreover, if $Lu = f$ in $\Omega$ for some elliptic operator $L$ with coefficients $a^{ij}, b^i, c^j \in C^\infty(\Omega)$ and some $f \in C^\infty(\Omega)$, then by the Sobolev embedding theorem $u \in C^\infty(\Omega)$.

The proof of interior regularity follows more or less from a difference quotient argument like before using induction on $k$ and energy estimates in place of the Schauder estimates in the case $k = 0$. However, we need to establish that the obvious difference quotient operator
\[ \delta_{l,h}u(x) = \frac{u(x + he_l) - u(x)}{h}, \tag{20} \]
where $h \neq 0$ and $l = 1, \ldots, n$, has the correct properties in the case that $u$ is a Sobolev function. We also need to be careful since $\delta_{l,h}f$ is not necessarily bounded locally in $W^{k,2}$ for $f \in W^{k,2}_{loc}(\Omega)$.

Lemma 2. Let $u \in W^{1,p}(\Omega)$ for $1 \leq p < \infty$. Then $\delta_{l,h}u \in L^p(\Omega')$ for any $\Omega' \subset \subset \Omega$ with $\text{dist}(\Omega', \partial \Omega) > h$ and
\[ \|\delta_{l,h}u\|_{L^p(\Omega')} \leq \|Du\|_{L^p(\Omega)}. \]

Proof. Since $C^\infty(\Omega)$ is dense in $W^{1,p}(\Omega)$ (see Gilbarg and Trudinger Theorem 7.9), it suffices to consider $u \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$. We compute
\[
\int_{\Omega'} |\delta_{l,h}u(x)|^p dx = \int_{\Omega'} \left| \frac{1}{h} \int_0^h D_l u(x + te_l) dt \right|^p dx \quad \text{(by the fundamental theorem of calculus)}
\leq \int_{\Omega'} \frac{1}{h} \int_0^h |D_l u(x + te_l)|^p dt dx \quad \text{(by Hölder’s inequality)}
\leq \frac{1}{h} \int_0^h \int_{\Omega'} |D_l u(x + te_l)|^p dx dt \quad \text{(by Tonelli’s theorem / Fubini’s theorem)}
\leq \frac{1}{h} \int_0^h \int_\Omega |D_l u(y)|^p dy dt \quad \text{(by letting } y = x + te_l)
\leq \int_{\Omega} |D_l u(y)|^p dy.
\]
Lemma 3. Let \( u \in L^p(\Omega) \) for \( 1 < p < \infty \) and suppose
\[
\sup_{0 < |h| < h_0} \| \delta_{l,h} u \|_{L^p(\Omega')} < \infty
\]
for every \( \Omega' \subset \subset \Omega \) and \( h_0 = \text{dist}(\Omega', \partial \Omega) \). Then the weak derivative \( D_l u \in L^p_{loc}(\Omega) \) exists and
\[
\| D_l u \|_{L^p(\Omega')} \leq \sup_{0 < |h| < h_0} \| \delta_{l,h} u \|_{L^p(\Omega')}
\]
for every \( \Omega' \subset \subset \Omega \) and \( h_0 = \text{dist}(\Omega', \partial \Omega) \).

Proof. Example sheet.

Proof of \( W^{2,2} \) Interior Regularity. Recall that
\[
\int_{\Omega} ((a^{ij} D_j u + b^i u) D_i \zeta - (c^j D_j u + d u) \zeta) = - \int_{\Omega} f \zeta
\]
for every \( \zeta \in W^{1,2}_0(\Omega) \). Since \( b^i \in C^{0,1}(\Omega) \) and \( u \in W^{1,2}(\Omega) \), by integration by parts,
\[
\int_{\Omega} a^{ij} D_j u D_i \zeta = \int_{\Omega} ((b^i + c^i) D_i u + (D_i b^i + d) u - f) \zeta = \int_{\Omega} g \zeta
\]
for every \( \zeta \in W^{1,2}_0(\Omega) \), where \( g = (b^i + c^i) D_i u + (D_i b^i + d) u - f \). Replace \( \zeta \) by \( \delta_{l,-h} \zeta \) to get
\[
\int_{\Omega} a^{ij} (x + h e_i) D_j \delta_{l,h} u(x) D_i \zeta(x) \, dx = \int_{\Omega} \delta_{l,h} (a^{ij} D_j u) D_i \zeta - \int_{\Omega} \delta_{l,h} a^{ij} D_j u D_i \zeta
\]
\[
= - \int_{\Omega} a^{ij} D_j u D_i \delta_{l,-h} \zeta - \int_{\Omega} \delta_{l,h} a^{ij} D_j u D_i \zeta
\]
\[
= - \int_{\Omega} g \delta_{l,-h} \zeta - \int_{\Omega} \delta_{l,h} a^{ij} D_j u D_i \zeta
\]
for every \( \zeta \in W^{1,2}_0(\Omega) \). Using (5), Cauchy-Schwartz, and the properties of difference quotients,
\[
\left| \int_{\Omega} a^{ij} (x + h e_i) D_j \delta_{l,h} u(x) D_i \zeta(x) \, dx \right| \leq \| g \|_{L^2(\Omega)} \| \delta_{l,-h} \zeta \|_{L^2(\Omega)} + C \| Du \|_{L^2(\Omega)} \| D \zeta \|_{L^2(\Omega)}
\]
\[
\leq C(\| g \|_{L^2(\Omega)} + \| Du \|_{L^2(\Omega)}) \| D \zeta \|_{L^2(\Omega)}
\]
for every \( \zeta \in W^{1,2}_0(\Omega) \), where \( \| g \|_{L^2} \) and \( \| Du \|_{L^2} \) are \( L^2 \) norms over the support of \( \zeta \), provided \( |h| \) is less than the distance of the support of \( \zeta \) to \( \partial \Omega \).

Choose \( \Omega' \subset \subset \Omega \) and \( \eta \in C_0^1(\Omega) \) satisfying \( 0 \leq \eta \leq 1 \) on \( \Omega \), \( \eta = 1 \) on \( \Omega' \) and \( |D \eta| \leq 2/d(\Omega', \Omega) \). Then, taking \( \zeta = \eta^2 \delta_{l,h} u \), for \( h \) sufficiently small (depending on the support of \( \eta \)) the previous computation, the ellipticity assumption, and the assumption that \( |\eta| \leq 1 \) imply that
\[
\lambda \int_{\Omega} \eta^2 |D \delta_{l,h} u|^2 \, dx \leq \int_{\Omega} a^{ij} (x + h e_i) \eta^2 D_j \delta_{l,h} u(x) D_i \delta_{l,h} u(x) \, dx
\]
\[
= \int_{\Omega} a^{ij} (x + h e_i) D_j \delta_{l,h} u(x) D_i (\eta^2 \delta_{l,h} u)(x) \, dx
\]
\[
- 2 \int_{\Omega} a^{ij} (x + h e_i) \eta \delta_{l,h} u(x) D_j \delta_{l,h} u(x) D_i \eta(x) \, dx
\]
\[
\leq C (1 + \| g \|_{L^2(\Omega)} + \| Du \|_{L^2(\Omega)}) (\| \eta D \delta_{l,h} u \|_{L^2(\Omega)} + \| (\delta_{l,h} u) D \eta \|_{L^2(\Omega)})
\]
Absorbing the $D\delta_{l,h}u$ terms into the right hand side and using the above lemmas to relate the discrete difference quotient to the derivative, we find that

$$\lambda \int_{\Omega'} |D\delta_{l,h}u|^2 dx \leq C \int_{\Omega} (|u|^2 + |Du|^2 + |g|^2) dx.$$  

Thus, because $\delta_{l,h}Du$ is uniformly bounded in $L^2(\Omega')$, we see that $u \in W^{2,2}(\Omega)$. Letting $h \to 0$, the estimate follows.

**Theorem 7** $(W^{k+2,2}$ Global Regularity). Let $k \geq 1$ be an integer. Let $\Omega$ be a $C^{k+2}$ domain in $\mathbb{R}^n$. Suppose $u \in W^{1,2}(\Omega)$ satisfies

$$Lu = D_i(a^{ij}D_ju + b^i u) + c^j D_ju + du = f \text{ weakly in } \Omega$$

for an elliptic operator $L$ with coefficients $a^{ij}, b^i, c^j, d \in C^{k,1}(\overline{\Omega})$, $f \in W^{k,2}(\Omega)$, and $\varphi \in W^{k+2,2}(\Omega)$. Then $u \in W^{k+2,2}(\Omega)$ with

$$\|u\|_{W^{2,2}(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|\varphi\|_{W^{2,2}(\Omega)})$$

if $k = 0$ and

$$\|u\|_{W^{k+2,2}(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{W^{k,2}(\Omega)} + \|\varphi\|_{W^{k+2,2}(\Omega)})$$

if $k \geq 1$ for some constant $C = C(n, k, L, \Omega) \in (0, \infty)$. Moreover, if $Lu = f$ in $\Omega$ and $u = \varphi$ on $\partial\Omega$ for some elliptic operator $L$ with coefficients $a^{ij}, b^i, c^j, d \in C^\infty(\overline{\Omega})$ and some $f, \varphi \in C^\infty(\Omega)$, then by the Sobolev embedding theorem $u \in C^\infty(\Omega)$.

The proof is fairly standard. We proceed by induction on $k$. To prove the $W^{2,2}$ regularity near a point $y \in \partial\Omega$, we can reduce to the case where $\varphi = 0$ by replacing $u$ with $u - \varphi$ and we can replace to the case where $y = 0$ and $\Omega \cap B_R(0) = B_R^+$ by using a $C^1$ diffeomorphism. By applying the difference quotient argument in the proof of $W^{2,2}$ interior regularity, using the fact that $\eta^2 u \in W_0^{1,2}(\Omega)$ when $\eta \in C^\infty(\mathbb{R}^n)$ is the cutoff function such that $\eta = 1$ on $B_{R^2/2}$, $\eta = 0$ on $\mathbb{R}^n \setminus B_R$, and $|D\eta| \leq 3/R$, we can show that $D_{ij}u \in W^{1,2}(B_{R^2/2}^+)$ for $l = 1, 2, \ldots, n - 1$. By the differential equation,

$$a^{ij}D_{ij}u = f - \sum_{(i,j) \neq (n,n)} a^{ij}D_{ij}u - \sum_{j=1}^n \left( \sum_{i=1}^n D_i(a^{ij} + b^i + c^j) + D_j(u - \sum_{i=1}^n D_ib^i + d) \right) \in L^2(B_{R^2/2}^+),$$

completing the proof that $u \in W^{2,2}(B_{R^2/2}^+)$. Note that as an immediate consequence of the existence theory and global regularity, whenever $\Omega$ is a $C^\infty$ domain, $a^{ij}, b^i, c^j, d \in C^\infty(\overline{\Omega})$ satisfy (4), (5), and (7), and $f \in C^\infty(\overline{\Omega})$, there exists a unique function $u \in C^\infty(\Omega)$ such that $Lu = f$ weakly in $\Omega$ and $u = \varphi$ on $\partial\Omega$. As was discussed previously, this implies that $Lu = f$ pointwise in $\Omega$ and $u = \varphi$ pointwise on $\partial\Omega$.

Note that by using the scaling argument from the proof of the $C^{2,\mu}$ Schauder estimates for classical solutions, we also get $C^{1,\mu}$ Schauder estimates on weak solutions to elliptic equations in divergence form. For example:

**Theorem 8** (Interior $C^{1,\mu}$ Estimate). Let $\mu \in (0, 1)$. Suppose $u \in C^{1,\mu}(\overline{B_R(x_0)})$ satisfies

$$Lu = D_i(a^{ij}D_ju + b^i u) + c^j D_ju + du = D_i f + g \text{ weakly in } B_R(x_0)$$
where
\[ a^{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2 \text{ for a.e. } x \in B_R(x_0) \text{ and all } \xi \in \mathbb{R}^n \]
for some constant \( \lambda > 0 \) and \( a^{ij}, b^i \in C^{0,\mu}(B_R(x_0)) \) and \( c^i, d \in C^0(B_R(x_0)) \) such that
\[
|a^{ij}|'_{0,\mu;B_R(x_0)} + R|b^i|'_{0,\mu;B_R(x_0)} + R|c^i|_{0;B_R(x_0)} + R^2|d|_{0;B_R(x_0)} \leq \nu
\]
for some constant \( \nu \in (0, \infty) \) and \( f^i \in C^{0,\mu}(\overline{B_R(x_0)}) \) and \( g \in C^0(B_R(x_0)) \). Then
\[
|u|'_{1,\mu;B_{R/2}(x_0)} \leq C(\|u\|_{L^2(B_R(x_0))} + R^{1+\mu}[f]_{\mu;B_R(x_0)} + R^2|g|_{0;B_R(x_0)})
\]
for some constant \( C = C(n, \lambda, \nu) \in (0, \infty) \).

References: Gilbarg and Trudinger, Chapter 8.