

INTRODUCTION TO NONLINEAR WAVE EQUATIONS

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1. INTRODUCTION

Warning: This is a first draft of the lecture notes and should be used with care!

In this course, we consider nonlinear wave equations. This can take the form of a scalar equation or a system of equations. A scalar wave equation takes the form

$$(1.1) \quad \sum_{\mu, \nu=0}^n \frac{1}{\sqrt{-\det g}} \partial_\mu (\sqrt{-\det g} (g^{-1})^{\mu\nu} \partial_\nu \phi) = F(\phi, \partial\phi),$$

where g is a Lorentzian metric on $I \times \mathbb{R}^n$ and the unknown function $\phi : I \times \mathbb{R}^n \rightarrow \mathbb{R}$, where I is a subinterval of \mathbb{R} .

We say that g is a Lorentzian metric if it is a symmetric $(n+1) \times (n+1)$ matrix (thus having real eigenvalues) with eigenvalues $\lambda_0 < 0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. It thus makes sense to divide by $\sqrt{-\det g}$ and also to define the inverse of g .

We say that the equation is **linear** if g is independent of ϕ and F is a linear function of both of its arguments; the equation is **semilinear** if g is independent of ϕ and F is a nonlinear function; and the equation is **quasilinear** if g is a function of ϕ and/or $\partial\phi$.

We will be mostly interested in the *Cauchy problem* for (1.1). Suppose we label the coordinate of the interval I by t or x^0 and the coordinates of \mathbb{R}^n by (x_1, \dots, x_n) and assume that g has the property that $g(\partial_{x_i}, \partial_{x_j})$ is a strictly positive definite matrix for $i, j = 1, \dots, n$. We can then pose data on the $\{t=0\}$ hypersurface, i.e., we prescribe $(\phi, \partial_t \phi)|_{\{t=0\}}$ to be some appropriate functions. The *Cauchy problem* asks for a solution to (1.1) such that the restrictions of ϕ and $\partial_t \phi$ to the hypersurface $\{t=0\}$ are precisely the prescribed functions.

More generally, we are also interested in system of equations. Consider $\phi : I \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfying the equation

$$(1.2) \quad \sum_{\mu, \nu=0}^n \frac{1}{\sqrt{-\det g_I}} \partial_\mu (\sqrt{-\det g_I} (g_I^{-1})^{\mu\nu} \partial_\nu \phi_I) = F_I(\phi, \partial\phi),$$

where $I = 1, \dots, m$.

Wave equations occurs naturally in many physical theories. We now look at a number of examples, many of which we will consider in the course.

Example 1.1 (Linear wave equation, D'Alembert 1749).

$$\square_m \phi = -\partial_t^2 \phi + \sum_{i=1}^m \partial_{x_i}^2 \phi = 0.$$

Example 1.2 (Maxwell's equation, Maxwell 1861-1862).

$$\begin{cases} \partial_t E = \nabla \times B \\ \partial_t B = -\nabla \times E \\ \nabla \cdot E = 0 \\ \nabla \cdot B = 0. \end{cases}$$

A priori, they may not look like wave equations (they are not even second order!). However, if we differentiate the first equation by ∂_t , use the second and third equations, we get

$$\square E_i = 0$$

for $i = 1, 2, 3$. Similarly,

$$\square B_i = 0.$$

Thus, given initial data $(E, B) \upharpoonright_{\{t=0\}} = (E^0, B^0)$ which are divergence-free, we can solve

$$\begin{cases} \square E_i = 0, & \square B_i = 0 \\ (E_i, \partial_t E_i) \upharpoonright_{\{t=0\}} = (E_i^0, (\nabla \times B^0)_i), & (B_i, \partial_t B_i) \upharpoonright_{\{t=0\}} = (B_i^0, -(\nabla \times E^0)_i). \end{cases}$$

Exercise: Show that the solution indeed satisfies the Maxwell equations.

Example 1.3 (Irrotational compressible fluids, Euler 1752). A fluid in $I \times \mathbb{R}^3$ is described by a vector field v describing the velocity of the fluid and a non-negative function h describing the enthalpy.

Define the pressure p to be a function of the enthalpy $p = p(h)$ such that

- (1) $p > 0$,
- (2) $\rho := \frac{dp}{dh} > 0$,
- (3) $\eta^2 := \rho \left(\frac{d^2 p}{dh^2} \right)^{-1} > 0$.

We call ρ the density of the fluid and η the speed of sound. The Euler equations are given by

$$(1.3) \quad \begin{cases} \frac{\partial}{\partial t} v_i + (v \cdot \nabla) v_i = -\frac{\partial h}{\partial x_i} \\ \frac{\partial}{\partial t} \rho + \nabla \cdot (\rho v) = 0, \end{cases}$$

for $i = 1, 2, 3$. We say that a flow is **irrotational** if $\nabla \times v = 0$. In that case, we can write $v = -\nabla \phi$, where ϕ is defined up to adding a function of time.

The first equation in (1.3) gives

$$\nabla \left(\frac{\partial \phi}{\partial t} - \frac{1}{2} |\nabla \phi|^2 - h \right) = 0.$$

Since we have the freedom to add a function of time to ϕ (which does not change v), we can choose

$$\frac{\partial \phi}{\partial t} - \frac{1}{2} |\nabla \phi|^2 - h = 0.$$

Then the second equation in (1.3) gives

$$\eta^{-2} \frac{\partial^2 \phi}{\partial t^2} - 2\eta^{-2} \frac{\partial \phi}{\partial x_i} \frac{\partial^2 \phi}{\partial t \partial x_i} + \eta^{-2} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} - \frac{\partial^2 \phi}{\partial x_i \partial x_i} = 0.$$

Here, we have use the convention that repeated indices are summed over.

Example 1.4 (Einstein vacuum equations, Einstein 1915). The Einstein vacuum equations describe the propagate of gravitational waves in the absence of matter and take the form

$$\text{Ric}(g) = 0$$

where the Lorentzian metric g is the unknown. In a coordinate system, these equations take the form (**Exercise**)

$$\text{Ric}(g)_{\mu\nu} = -\frac{1}{2}(g^{-1})^{\alpha\beta}\partial_{\alpha\beta}^2 g_{\mu\nu} - \frac{1}{2}(g^{-1})^{\alpha\beta}\partial_{\mu\nu}^2 g_{\alpha\beta} + \frac{1}{2}(g^{-1})^{\alpha\beta}\partial_{\alpha\nu}^2 g_{\beta\mu} + \frac{1}{2}(g^{-1})^{\alpha\beta}\partial_{\beta\mu}^2 g_{\alpha\nu} + F_{\mu\nu}(g, \partial g),$$

where $F_{\mu\nu}(g, \partial g)$ is a function of g and its derivatives. Here, we summed over repeated indices and use g to denote the determinant of g .

This does not look like a wave equation (because of the second to fourth terms)! However, as we will see later in the course that a more careful choice of coordinates allows one to rewrite this system as a system of nonlinear wave equations.

Example 1.5 (Wave map equations). Let $\phi : I \times \mathbb{R}^n \rightarrow \mathbb{S}^m = \{x \in \mathbb{R}^{m+1} : |x| = 1\}$. The wave map equation is given by

$$\square\phi = -\phi(\partial_t\phi^t\partial_t\phi - \sum_{i=1}^n \partial_i\phi^i\partial_i\phi).$$

Exercise: Show that this is well-defined, i.e., suppose that $|\phi_0|^2 = 1$ and $\phi_0^t\phi_1 = 0$. Then, if a solution ϕ exists in $I \times \mathbb{R}^n$, then $|\phi|^2 = 1$, i.e., ϕ is indeed a map to the sphere.

We will begin the course by studying the linear wave equation. As we will see, the solution for the Cauchy problem for the linear wave equation can in fact be written down explicitly. Nevertheless, we will single out 3 + 1 properties here:

- (0) Existence and uniqueness of solutions.
- (1) (Conservation of energy)

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^n} ((\partial_t\phi)^2 + \sum_{i=1}^n (\partial_i\phi)^2)(t, x) dx.$$

is independent of time.

- (2) (Dispersion) The solution ϕ satisfies a decay estimate

$$\sup_x |\phi|(t, x) \leq C(1+t)^{-\frac{n-1}{2}}$$

for some $C > 0$. (One may immediately ask: how can conservation and decay happen at the same time? The answer is that the support of the function grows and therefore even the pointwise value of ϕ is decaying, the L^2 norm is conserved. This is why the phenomenon is called *dispersion*.)

- (3) (Finite speed of propagation) If $(\phi_0, \phi_1) = (0, 0)$ in $\{y \in \mathbb{R}^n : |y - x| \leq t\}$, then $\phi(t, x) = 0$.

On the other hand, for nonlinear equations, things are very different!

- (0) One has a *general* theory for existence and uniqueness of *local* solutions.
- (1) There is a wide range of large time behaviour:
 - (a) Some solutions (to some equations) behave like linear equation and disperse
 - (b) Some are global-in-time but do not disperse
 - (c) Some are not global-in-time
- (2) Restricting to small and localized initial data, the solutions behave like their linear counterpart for a long time. However, even in this case, there can be very different global behaviour.

2. LINEAR WAVE EQUATION VIA FOURIER TRANSFORM

We use the Fourier transform to study the wave equation

$$\begin{cases} \square\phi = 0 \\ (\phi, \partial_t\phi)|_{\{t=0\}} = (\phi_0, \phi_1). \end{cases}$$

Taking the Fourier transform of the wave equation and noting that

$$\mathcal{F}(\partial_i\phi)(\xi) = 2\pi i\xi_i\hat{\phi}(xi),$$

we get

$$-\partial_t^2 \hat{\phi}(\xi) - 4\pi^2 |\xi|^2 \hat{\phi}(\xi) = 0.$$

The general solution to this ODE is given by

$$\hat{\phi}(t, \xi) = A(\xi) \sin(2\pi t|\xi|) + B(\xi) \cos(2\pi t|\xi|).$$

The initial conditions imply that

$$A(\xi) = \frac{\hat{\phi}_1(\xi)}{2\pi|\xi|}, \quad B(\xi) = \hat{\phi}_0(\xi),$$

i.e.

$$\hat{\phi}(t, \xi) = \frac{\hat{\phi}_1(\xi)}{2\pi|\xi|} \sin(2\pi t|\xi|) + \hat{\phi}_0(\xi) \cos(2\pi t|\xi|).$$

We have thus shown the existence of a solution! In fact, as long as we require the solution to be sufficiently regular such that all the above operations make sense, this is in fact the unique solution. At the very least, it is easy to see that this is the unique solution such that

$$(\phi, \partial_t \phi) \in L^\infty([0, T]; H^1(\mathbb{R}^n)) \times L^\infty([0, T]; L^2(\mathbb{R}^n)).$$

We will now show the properties of the solution. First, the conservation of energy:

Proposition 2.1.

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^n} ((\partial_t \phi)^2 + \sum_{i=1}^n (\partial_i \phi)^2)(t, x) dx$$

is independent of time.

Proof. By Plancherel Theorem,

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^n} ((\partial_t \hat{\phi})^2 + \sum_{i=1}^n (2\pi \xi_i \hat{\phi})^2)(t, \xi) d\xi.$$

We shall show that in fact the integrand is independent of time. First,

$$|\partial_t \hat{\phi}|^2 = \left(\hat{\phi}_1(\xi) \cos(2\pi t|\xi|) - \hat{\phi}_0(\xi) 2\pi|\xi| \sin(2\pi t|\xi|) \right)^2$$

Also,

$$\begin{aligned} \sum_{i=1}^n (2\pi \xi_i \hat{\phi})^2 &= 4\pi^2 |\xi|^2 \left(\frac{\hat{\phi}_1(\xi)}{2\pi|\xi|} \sin(2\pi t|\xi|) + \hat{\phi}_0(\xi) \cos(2\pi t|\xi|) \right)^2 \\ &= |\hat{\phi}_1(\xi)|^2 \sin^2(2\pi|\xi|t) + 4\pi^2 |\xi|^2 |\hat{\phi}_0(\xi)|^2 \cos^2(2\pi|\xi|t) + 4\pi|\xi| \hat{\phi}_0(\xi) \hat{\phi}_1(\xi) \sin(2\pi t|\xi|) \cos(2\pi t|\xi|). \end{aligned}$$

Summing, we get

$$|\partial_t \hat{\phi}|^2 + \sum_{i=1}^n (2\pi \xi_i \hat{\phi})^2 = |\hat{\phi}_1(\xi)|^2 + 4\pi^2 |\xi|^2 |\hat{\phi}_0(\xi)|^2,$$

which is manifestly independent of t . □

We then look at the decay of ϕ in t . Why do we expect the solution to decay¹? The reason is that as t becomes large, there are more oscillations in $\sin(2\pi t|\xi|)$ and $\cos(2\pi t|\xi|)$. Before we proceed, we first rewrite the solution:

$$(2.1) \quad \hat{\phi}(t, \xi) = e^{2\pi i t|\xi|} \left(\frac{\hat{\phi}_0(\xi)}{2} + \frac{\hat{\phi}_1(\xi)}{2\pi i|\xi|} \right) + e^{-2\pi i t|\xi|} \left(\frac{\hat{\phi}_0(\xi)}{2} - \frac{\hat{\phi}_1(\xi)}{2\pi i|\xi|} \right).$$

We first prove a slightly simpler decay result which hold in the region $\{|x| \leq R\}$, but the estimate degenerates as R becomes large.

¹Of course we argued previously that this is due to dispersion in physical space, but we want to see this in Fourier variables.

Proposition 2.2. *Fix any $R > 0$. There exists $C = C(\phi_0, \phi_1, R)$ such that*

$$|\phi(t, x)| \leq \frac{C}{(1+t)^{n-1}}$$

whenever $|x| \leq R$.

Proof. We will only consider the case $n = 3$ and the integral

$$I = \int_{\mathbb{R}^3} e^{2\pi i(t|\xi|+x \cdot \xi)} \frac{\hat{\phi}_1(\xi)}{2\pi i|\xi|} d\xi.$$

(Of course, according to the (2.1), there are four such terms to consider.) We make a further simplification that we only consider $t \geq 1$, since it is clear that $|\phi(t, x)| \leq C$ using the fact that $\hat{\phi}_1(\xi) \in \mathcal{S}$.

To proceed, we rewrite ξ in polar coordinates $(|\xi|, \xi_\theta, \xi_\varphi)$ defined by the relations

$$|\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2, \quad \xi_3 = |\xi| \cos \xi_\theta, \quad \xi_1 = |\xi| \sin \xi_\theta \cos \xi_\varphi.$$

In this coordinate system, the volume form is $d\xi = |\xi|^2 \sin \xi_\theta d|\xi| d\xi_\theta d\xi_\varphi$.

The key point is to notice that

$$\left(\frac{1}{2\pi i t} \frac{\partial}{\partial |\xi|} \right)^N e^{2\pi i t |\xi|} = e^{2\pi i t |\xi|}$$

for every non-negative integer N . Using this, we can integrate by parts in $|\xi|$ to get

$$\begin{aligned} I &= - \int_{\mathbb{R}^3} e^{2\pi i t |\xi|} \frac{1}{2\pi i t} \frac{\partial}{\partial |\xi|} \left(e^{2\pi i x \cdot \xi} \frac{\hat{\phi}_1(\xi)}{2\pi i} |\xi| \right) \sin \xi_\theta d|\xi| d\xi_\theta d\xi_\varphi \\ &= \int_{\mathbb{R}^3} e^{2\pi i t |\xi|} \frac{1}{4\pi^2 t} e^{2\pi i x \cdot \xi} \left((2\pi i x \cdot \xi) \hat{\phi}_1(\xi) + \hat{\phi}_1(\xi) + |\xi| \frac{\partial}{\partial |\xi|} \hat{\phi}_1(\xi) \right) \sin \xi_\theta d|\xi| d\xi_\theta d\xi_\varphi \end{aligned}$$

Notice that there are no boundary terms arising from the integration by parts since $\hat{\phi}_1(\xi)$ is in Schwartz class and $|\xi| \hat{\phi}_1(\xi)$ vanishes at $|\xi| = 0$. Note that we have thus gained a power of $t!$ We integrate by parts one more time to gain another power of t . Notice that this time we have a boundary term at $|\xi| = 0$:

$$\begin{aligned} I &= - \int_{\mathbb{R}^3} e^{2\pi i t |\xi|} \frac{1}{8\pi^3 i t^2} \frac{\partial}{\partial |\xi|} \left(e^{2\pi i x \cdot \xi} \left((2\pi i x \cdot \xi) \hat{\phi}_1(\xi) + \hat{\phi}_1(\xi) + |\xi| \frac{\partial}{\partial |\xi|} \hat{\phi}_1(\xi) \right) \right) \sin \xi_\theta d|\xi| d\xi_\theta d\xi_\varphi \\ &\quad + \frac{1}{2\pi^2 i t^2} \hat{\phi}_1(\xi = 0) \\ &= - \int_{\mathbb{R}^3} e^{2\pi i t |\xi|} \frac{1}{8\pi^3 i t^2} e^{2\pi i x \cdot \xi} \left((2\pi i x \cdot \xi)^2 \hat{\phi}_1(\xi) + \frac{2\pi i x \cdot \xi}{|\xi|} \hat{\phi}_1(\xi) + (4\pi i x \cdot \xi) \frac{\partial}{\partial |\xi|} \hat{\phi}_1(\xi) \right. \\ &\quad \left. + \frac{\partial}{\partial |\xi|} \hat{\phi}_1(\xi) + \frac{\partial}{\partial |\xi|} (|\xi| \frac{\partial}{\partial |\xi|} \hat{\phi}_1(\xi)) \right) \sin \xi_\theta d|\xi| d\xi_\theta d\xi_\varphi + \frac{1}{2\pi^2 i t^2} \hat{\phi}_1(\xi = 0). \end{aligned}$$

The boundary term in particular does not allow us to integrate by parts again. We thus simply bound each of these terms. Using the fact the $\hat{\phi}_1 \in \mathcal{S}$, it is easy to see that

$$|I| \leq \frac{C(1+R^2)}{t^2},$$

as desired. □

Of course, the above decay bound is not uniform for all $x \in \mathbb{R}^n$. The following theorem, on the other hand, gives a uniform decay rate. Notice that the decay rate obtained is weaker²:

Theorem 2.3. *There exists $C = C(\phi_0, \phi_1)$ such that*

$$|\phi(t, x)| \leq \frac{C}{(1+t)^{\frac{n-1}{2}}}.$$

²Nevertheless, as we will see in the Example Sheet, it is sharp!

Proof. We give a proof that works when n is odd. As before, we will again only consider $n = 3$ and the term

$$I = \int_{\mathbb{R}^3} e^{2\pi i(t|\xi|+x\cdot\xi)} \frac{\hat{\phi}_1(\xi)}{2\pi i|\xi|} d\xi.$$

Without loss of generality, we assume that $x = (0, 0, \alpha)$ for some $\alpha \in \mathbb{R}$. Let $\rho = \sqrt{\xi_1^2 + \xi_2^2}$. We use the coordinate system $(\xi_3, \rho, \xi_\theta)$, where ξ_θ is given by

$$\rho \cos \xi_\theta = \xi_1.$$

We now try to repeat the argument in the proof of Proposition 2.2 with $\frac{\partial}{\partial|\xi|}$ replaced by $\frac{\partial}{\partial\rho}$. The key point is of course that $\frac{\partial}{\partial\rho} e^{2\pi i x \cdot \xi} = 0$ so we do not get a dependence on x . We first note that

$$\frac{\partial|\xi|}{\partial\rho} = \frac{\rho}{|\xi|}$$

and thus for every non-negative integer N , we have

$$\left(\frac{|\xi|}{2\pi i \rho t} \frac{\partial}{\partial\rho}\right)^N e^{2\pi i t|\xi|} = e^{2\pi i t|\xi|}.$$

Using this, we can integrate by parts in ρ . Notice that the volume form is $d\xi = \rho d\rho d\xi_\theta d\xi_3$. We thus have

$$I = - \int_{\mathbb{R}^3} e^{2\pi i t|\xi|+2\pi i x \cdot \xi} \frac{1}{2\pi i t} \frac{\partial}{\partial\rho} \left(\frac{\hat{\phi}_1(\xi)}{2\pi i} \right) d\rho d\xi_\theta d\xi_3 - \int e^{2\pi i t|\xi_3|+2\pi i \alpha \xi_3} \frac{1}{it} \frac{\hat{\phi}_1(\rho=0, \xi_3)}{2\pi i} d\xi_3,$$

where the last term is a boundary term at $\rho = 0$. It is easy to see that both terms are bounded in magnitude by $\frac{C}{t}$, uniformly in x . \square

Remark 2.4. The proof we gave above would not give the desired result when n is even. In fact, in that case one only obtains the weaker decay

$$|\phi(t, x)| \leq \frac{C}{(1+t)^{\frac{n-2}{2}}}.$$

Exercise: Check this.

In view of the above remark, we give another proof of the theorem. In particular, this gives a more refined bound away from the set $\{(t, x) : (1-\epsilon)t \leq |x| \leq (1+\epsilon)t\}$.

Proof. (0) As before, we will again only consider $n = 3$ and the term

$$I = \int_{\mathbb{R}^3} e^{2\pi i(t|\xi|+x\cdot\xi)} \frac{\hat{\phi}_1(\xi)}{2\pi i|\xi|} d\xi.$$

Again, without loss of generality, we assume that $x = (0, 0, \alpha)$ for some $\alpha \in \mathbb{R}$. Let $\rho = \sqrt{\xi_1^2 + \xi_2^2}$. We use the coordinate system $(\xi_3, \rho, \xi_\theta)$, where ξ_θ is given by

$$\rho \cos \xi_\theta = \xi_1.$$

Introduce a cutoff function depending only on $\frac{\xi_3^2}{\xi_3^2 + \rho^2}$ such that

$$(2.2) \quad \chi(\xi) = \begin{cases} 1, & \text{if } \xi_3^2 < (1-2\epsilon^2)|\xi|^2 \\ 0, & \text{if } \xi_3^2 \geq (1-\epsilon^2)|\xi|^2. \end{cases}$$

We then consider

$$I = I_1 + I_2 := \int_{\mathbb{R}^3} e^{2\pi i(t|\xi|+x\cdot\xi)} \chi(\xi) \frac{\hat{\phi}_1(\xi)}{2\pi i|\xi|} d\xi + \int_{\mathbb{R}^3} e^{2\pi i(t|\xi|+x\cdot\xi)} (1-\chi(\xi)) \frac{\hat{\phi}_1(\xi)}{2\pi i|\xi|} d\xi.$$

(1) We first bound I_1 . The key point is that in the support of χ , we have

$$\xi_3^2 < (1-\epsilon^2)|\xi|^2 \implies \xi_3^2 < \frac{1-\epsilon^2}{\epsilon^2} \rho^2.$$

Therefore,

$$\frac{\partial|\xi|}{\partial\rho} = \frac{\rho}{\sqrt{\rho^2 + \xi_3^2}}$$

is uniformly bounded below by a constant (depending on ϵ). Again, we want to integrate by parts in ρ using the fact that $\frac{\partial}{\partial \rho} e^{2\pi i x \cdot \xi} = 0$. We now turn to the details: We first note that for every non-negative integer N , we have

$$\left(\frac{|\xi|}{2\pi i \rho t} \frac{\partial}{\partial \rho}\right)^N e^{2\pi i t |\xi|} = e^{2\pi i t |\xi|}.$$

Using this, we can integrate by parts in ρ . Notice that the volume form is $d\xi = \rho d\rho d\xi_\theta d\xi_3$. We thus have

$$I_1 = - \int_{\mathbb{R}^3} e^{2\pi i t |\xi| + 2\pi i x \cdot \xi} \frac{1}{2\pi i t} \frac{\partial}{\partial \rho} \left(\frac{\chi(\xi) \hat{\phi}_1(\xi)}{2\pi i} \right) d\rho d\xi_\theta d\xi_3.$$

Notice that we do not have a boundary term at $\rho = 0$ since $\chi(\xi)$ is supported away from $\rho = 0$ for almost every ξ_3 . We can integrate by parts once more to get

$$I_1 = - \int_{\mathbb{R}^3} e^{2\pi i t |\xi| + 2\pi i x \cdot \xi} \frac{1}{4\pi^2 t^2} \frac{\partial}{\partial \rho} \left(\frac{|\xi|}{\rho} \frac{\partial}{\partial \rho} \left(\frac{\chi(\xi) \hat{\phi}_1(\xi)}{2\pi i} \right) \right) d\rho d\xi_\theta d\xi_3.$$

Notice that on the support of χ , we have an upper bound for $\frac{|\xi|}{\rho}$. This term is therefore bounded by $\frac{C}{t^2}$.

- (2) The difficult term is of course I_2 . We first make an observation that if $|\frac{\alpha}{t}| \leq \sqrt{1-\delta}$ or $|\frac{\alpha}{t}| \geq \frac{\sqrt{1+\delta}}{1-2\epsilon^2}$, then

$$\left(\frac{\alpha \xi_3}{t}\right)^2 \leq (1-\delta)|\xi|^2 \quad \text{or} \quad \left(\frac{\alpha \xi_3}{t}\right)^2 \geq (1+\delta)|\xi|^2$$

in the support of $1 - \chi(\xi)$. In other words, there exists a constant $C = C(\delta, \epsilon) > 0$ such that

$$(2.3) \quad \left| 1 + \frac{\alpha \xi_3}{t|\xi|} \right| \geq C^{-1} > 0.$$

We can therefore use $\frac{\partial}{\partial |\xi|}$ in the polar coordinates $(|\xi|, \xi_\theta, \xi_\varphi)$ defined as before by the relations

$$|\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2, \quad \xi_3 = |\xi| \cos \xi_\theta, \quad \xi_1 = |\xi| \sin \xi_\theta \cos \xi_\varphi.$$

Now, for every non-negative integer N , we write

$$\left(\frac{|\xi|}{2\pi i t (|\xi| + \frac{\alpha \xi_3}{t})} \frac{\partial}{\partial |\xi|}\right)^N e^{2\pi i (t|\xi| + \alpha \xi_3)} = e^{2\pi i (t|\xi| + \alpha \xi_3)}.$$

We now proceed in a similar way as Proposition 2.2. We can integrate by parts in $|\xi|$ to get

$$I_2 = \int_{\mathbb{R}^3} e^{2\pi i t |\xi|} e^{2\pi i x \cdot \xi} \frac{1}{4\pi^2 t} \frac{\partial}{\partial |\xi|} \left(\frac{|\xi|^2 (1 - \chi(\xi)) \hat{\phi}_1(\xi)}{(|\xi| + \frac{\alpha \xi_3}{t})} \right) \sin \xi_\theta d|\xi| d\xi_\theta d\xi_\varphi.$$

Notice that there are no boundary terms arising from the integration by parts since $\hat{\phi}_1(\xi)$ is in Schwartz class and on the support of $(1 - \chi)$, we have $\frac{|\xi|^2 (1 - \chi(\xi)) \hat{\phi}_1(\xi)}{2\pi i (|\xi| + \frac{\alpha \xi_3}{t})} \leq C(\delta, \epsilon) |\xi|$ so that the boundary term at $|\xi| = 0$ vanishes. We integrate by parts once more - noticing that this time we pick up a boundary term at $|\xi| = 0$:

$$\begin{aligned} I_2 = & - \int_{\mathbb{R}^3} e^{2\pi i t |\xi|} e^{2\pi i x \cdot \xi} \frac{1}{8\pi^3 i t^2} \frac{\partial}{\partial |\xi|} \left(\frac{|\xi|}{|\xi| + \frac{\alpha \xi_3}{t}} \frac{\partial}{\partial |\xi|} \left(\frac{|\xi|^2 (1 - \chi(\xi)) \hat{\phi}_1(\xi)}{(|\xi| + \frac{\alpha \xi_3}{t})} \right) \right) \sin \xi_\theta d|\xi| d\xi_\theta d\xi_\varphi \\ & + \int_{\mathbb{S}^2} \frac{1}{8\pi^3 i t^2} \lim_{|\xi| \rightarrow 0} \left(\frac{|\xi|}{|\xi| + \frac{\alpha \xi_3}{t}} \frac{\partial}{\partial |\xi|} \left(\frac{|\xi|^2 (1 - \chi(\xi)) \hat{\phi}_1(\xi)}{(|\xi| + \frac{\alpha \xi_3}{t})} \right) \right) \sin \xi_\theta d\xi_\theta d\xi_\varphi. \end{aligned}$$

Using (2.3), we in particular see that the last term is (in general non-zero and) bounded by $\frac{C}{t^2}$. Likewise, one can check using (2.3) that the first term is also bounded above by $\frac{C}{t^2}$. Therefore, if α and t lie in the range $|\frac{\alpha}{t}| \leq \sqrt{1-\delta}$ or $|\frac{\alpha}{t}| \geq \frac{\sqrt{1+\delta}}{1-2\epsilon^2}$, we in fact have a better decay estimate.

- (3) It remains to estimate I_2 when $|\frac{\alpha}{t}|$ is close to 1 - more precisely, when $\sqrt{1-\delta} < |\frac{\alpha}{t}| < \frac{\sqrt{1+\delta}}{1-2\epsilon^2}$. In this case, it suffices to prove decay in $|\alpha|$. We now use the coordinate system $(|\xi|, \xi_\theta, \xi_\varphi)$ as in part (2) of the proof. The phase function can be re-written (using $x = (0, 0, \alpha)$) as

$$2\pi i(t|\xi| + x \cdot \xi) = 2\pi i\alpha|\xi|(1 + \cos \xi_\theta) + 2\pi i|\xi|(t - \alpha).$$

Hence, we re-write I_2 as

$$I_2 = \int_{\mathbb{R}^3} e^{2\pi i\alpha|\xi|(1+\cos \xi_\theta)} e^{2\pi i|\xi|(t-\alpha)} (1 - \chi(\xi)) \frac{\hat{\phi}_1(\xi)}{2\pi i|\xi|} |\xi|^2 \sin \xi_\theta d|\xi| d\xi_\theta d\xi_\varphi.$$

We first take the integral in $d\xi_\theta$, noting in particular that $e^{2\pi i|\xi|(t-\alpha)}$ is independent of ξ_θ . In other words, we consider

$$J_2 = \int_0^\pi e^{2\pi i\alpha|\xi|(1+\cos \xi_\theta)} (1 - \chi(\xi)) \frac{\hat{\phi}_1(\xi)}{2\pi i} |\xi| \sin \xi_\theta d\xi_\theta.$$

Note that

$$|I_2| \leq \int_0^{2\pi} \int_0^\infty |J_2| d|\xi| d\xi_\varphi$$

and it suffices to get a bound on J_2 . Note that for every non-negative integer N

$$\left(\frac{-1}{2\pi i\alpha|\xi| \sin \xi_\theta} \frac{\partial}{\partial \xi_\theta} \right)^N e^{2\pi i\alpha|\xi|(1+\cos \xi_\theta)} = e^{2\pi i\alpha|\xi|(1+\cos \xi_\theta)}.$$

Integrating by parts in ξ_θ , we get

$$\begin{aligned} J_2 &= - \int_0^\pi \frac{e^{2\pi i\alpha|\xi|(1+\cos \xi_\theta)}}{4\pi^2\alpha} \frac{\partial}{\partial \xi_\theta} \left((1 - \chi(\xi)) \hat{\phi}_1(\xi) \right) d\xi_\theta \\ &\quad + \left(\frac{e^{2\pi i\alpha|\xi|(1+\cos \xi_\theta)}}{4\pi^2\alpha} (1 - \chi(\xi)) \hat{\phi}_1(\xi) \right) \Big|_{\{\xi_\theta=0\}}. \end{aligned}$$

This implies

$$|J_2| \leq \frac{C_M}{|\alpha|(1+|\xi|)^M}$$

for every integer M (since $\hat{\phi}_1 \in \mathcal{S}$). The result follows.

- (4) In the case that n is even, the proof of part (3) requires a modification. We consider the $n = 2$ case. Here, we assume $x = (0, \alpha)$ and use the polar coordinates $(|\xi|, \xi_\theta)$. The corresponding integral is

$$I_2 = \int_{\mathbb{R}^2} e^{2\pi i\alpha|\xi|(1+\cos \xi_\theta)} e^{2\pi i|\xi|(t-\alpha)} (1 - \chi(\xi)) \frac{\hat{\phi}_1(\xi)}{2\pi i} d|\xi| d\xi_\theta.$$

As in the $n = 3$ case, we introduce J_2 as the ξ_θ integral, i.e.,

$$J_2 = \int_0^{2\pi} e^{2\pi i\alpha|\xi|(1+\cos \xi_\theta)} (1 - \chi(\xi)) \frac{\hat{\phi}_1(\xi)}{2\pi i} d\xi_\theta.$$

Divide the integral into two parts

$$J_2 := J_{2,1} + J_{2,2} := \left(\int_0^\delta + \int_{\pi-\delta}^{\pi+\delta} + \int_{2\pi-\delta}^{2\pi} \right) + \int_{\{\xi_\theta \in [\delta, \pi-\delta] \cup [\pi+\delta, 2\pi-\delta]\}},$$

where δ is some small constant to be chosen. For $J_{2,1}$, we bound it using the smallness of the interval, i.e.,

$$|J_{2,1}| \leq \frac{C_M \delta}{(1+|\xi|)^M}$$

for every integer M . For $J_{2,2}$, we use as in the $n = 3$ case the identity

$$\left(\frac{-1}{2\pi i\alpha|\xi| \sin \xi_\theta} \frac{\partial}{\partial \xi_\theta} \right) e^{2\pi i\alpha|\xi|(1+\cos \xi_\theta)} = e^{2\pi i\alpha|\xi|(1+\cos \xi_\theta)}.$$

We thus get, after integrating by parts in ξ_θ ,

$$\begin{aligned}
 & J_{2,1} \\
 = & - \int_{\{\xi_\theta \in [\delta, \pi - \delta] \cup [\pi + \delta, 2\pi - \delta]\}} \frac{e^{2\pi i \alpha |\xi| (1 + \cos \xi_\theta)}}{4\pi^2 \alpha |\xi|} \frac{\partial}{\partial \xi_\theta} \left(\frac{(1 - \chi(\xi)) \hat{\phi}_1(\xi)}{\sin \xi_\theta} \right) d\xi_\theta \\
 & + \left(\frac{e^{2\pi i \alpha |\xi| (1 + \cos \xi_\theta)}}{4\pi^2 \alpha |\xi|} \frac{(1 - \chi(\xi)) \hat{\phi}_1(\xi)}{\sin \xi_\theta} \right) \Big|_{\{\xi_\theta = \pi - \delta\}} + \left(\frac{e^{2\pi i \alpha |\xi| (1 + \cos \xi_\theta)}}{4\pi^2 \alpha |\xi|} \frac{(1 - \chi(\xi)) \hat{\phi}_1(\xi)}{\sin \xi_\theta} \right) \Big|_{\{\xi_\theta = 2\pi - \delta\}} \\
 & - \left(\frac{e^{2\pi i \alpha |\xi| (1 + \cos \xi_\theta)}}{4\pi^2 \alpha |\xi|} \frac{(1 - \chi(\xi)) \hat{\phi}_1(\xi)}{\sin \xi_\theta} \right) \Big|_{\{\xi_\theta = \delta\}} - \left(\frac{e^{2\pi i \alpha |\xi| (1 + \cos \xi_\theta)}}{4\pi^2 \alpha |\xi|} \frac{(1 - \chi(\xi)) \hat{\phi}_1(\xi)}{\sin \xi_\theta} \right) \Big|_{\{\xi_\theta = \pi + \delta\}}.
 \end{aligned}$$

Clearly, all the boundary terms pick up a δ^{-1} from $\frac{1}{\sin \xi_\theta}$ and are bounded by

$$\frac{C_M \delta^{-1}}{|\alpha| |\xi| (1 + |\xi|)^M}.$$

On the other hand, for the first integral, we have

$$\begin{aligned}
 & \left| \int_{\{\xi_\theta \in [\delta, \pi - \delta] \cup [\pi + \delta, 2\pi - \delta]\}} \frac{e^{2\pi i \alpha |\xi| (1 + \cos \xi_\theta)}}{4\pi^2 \alpha |\xi|} \frac{\partial}{\partial \xi_\theta} \left(\frac{(1 - \chi(\xi)) \hat{\phi}_1(\xi)}{\sin \xi_\theta} \right) d\xi_\theta \right| \\
 & \leq \frac{C_M}{\alpha |\xi| (1 + |\xi|)^M} \int_{\{\xi_\theta \in [\delta, \pi - \delta] \cup [\pi + \delta, 2\pi - \delta]\}} \frac{1}{\sin^2 \xi_\theta} d\xi_\theta \\
 & \leq \frac{C_M \delta^{-1}}{|\alpha| |\xi| (1 + |\xi|)^M}.
 \end{aligned}$$

Therefore,

$$|J_2| \leq \frac{C_M}{(1 + |\xi|)^M} \left(\delta + \frac{\delta^{-1}}{|\alpha| |\xi|} \right).$$

Choosing $\delta = |\alpha|^{-\frac{1}{2}} |\xi|^{-\frac{1}{2}}$, we get

$$|J_2| \leq \frac{C_M}{|\alpha|^{\frac{1}{2}} |\xi|^{\frac{1}{2}} (1 + |\xi|)^M}.$$

This in turn implies

$$|I_2| \leq \frac{C}{|\alpha|^{\frac{1}{2}}} \leq \frac{C}{t^{\frac{1}{2}}}$$

as desired. □

Let us explicitly state the more refined bound away from the region $\{(t, x) : (1 - \epsilon)t \leq |x| \leq (1 + \epsilon)t\}$, which we derived in parts (1) and (2) of the proof above and in fact holds in all dimensions:

Corollary 2.5. *Given $\epsilon > 0$, define $S_t := \{x : |x| < (1 - \epsilon)t \text{ or } |x| > (1 + \epsilon)t\}$. Then, there exists $C = C(\phi_0, \phi_1, \epsilon) > 0$ such that*

$$\sup_{x \in S_t} |\phi|(t, x) \leq \frac{C}{(1 + t)^{n-1}}.$$

Remark 2.6. Of course, Corollary 2.5 implies Proposition 2.2!

3. LINEAR WAVE EQUATION VIA FUNDAMENTAL SOLUTION AND REPRESENTATION FORMULA

This section will be provided later.

4. ENERGY ESTIMATES

In the previous lectures, we studied the constant coefficient linear wave equation

$$(4.1) \quad \square\phi = 0$$

by solving the equation explicitly either via the Fourier transform or via the fundamental solution. In particular, we have shown the existence and uniqueness of solutions, conservation of energy, dispersion and the finite speed of propagation. We have also seen that depending on the property that we wanted to prove, one representation of the solution may be preferable to the other.

In this section, we will introduce yet another method to study the equation (4.1), namely the energy methods. In fact, we will show that we can prove (almost) all the properties of the solutions we previously showed using the energy methods. More importantly, as we will see, these methods are more robust and can deal with situations when the solutions to the equation may not be explicitly available. In particular, they will make a crucial appearance in the study of nonlinear equations!

We will use the term “energy methods” to refer loosely to methods which are based on L^2 type estimates. The most basic example is the conservation of energy:

Proposition 4.1. *Let ϕ be a (sufficiently regular) solution to (4.1). Define $E(t)$ as before as*

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^n} ((\partial_t\phi)^2 + \sum_{i=1}^n (\partial_i\phi)^2)(t, x) dx.$$

Then E is constant in time.

Proof. Consider

$$\begin{aligned} \frac{d}{dt} E(t) &= \int_{\mathbb{R}^n} \partial_t\phi \partial_t^2\phi + \sum_{i=1}^n \partial_t\partial_i\phi \partial_i\phi \\ &= \int_{\mathbb{R}^n} \partial_t\phi \sum_{i=1}^n \partial_i^2\phi + \sum_{i=1}^n \partial_t\partial_i\phi \partial_i\phi \\ &= \int_{\mathbb{R}^n} \partial_t\phi \sum_{i=1}^n \partial_i^2\phi - \sum_{i=1}^n \partial_t\phi \partial_i^2\phi = 0. \end{aligned}$$

Here, we have used the equation in the second line and have integrated by parts in the third line. □

Remark 4.2. Another (completely equivalent) way to look at the proof is to use the identity

$$0 = \partial_t\phi \square\phi$$

and integrate it in the region $[0, t] \times \mathbb{R}^n$ with respect to the volume form $dxdt$ and integrate by parts. We will take this point of view in some of the results below.

One immediate consequence of the energy identity Proposition 4.1 is the uniqueness of solutions:

Corollary 4.3. *Given initial data*

$$(\phi, \partial_t\phi) \upharpoonright_{\{t=0\}} = (\phi_0, \phi_1),$$

if we have two solutions $\phi^{(1)}$ and $\phi^{(2)}$ to the linear wave equation with the given data such that both of them satisfy $(\phi^{(i)}, \partial_t\phi^{(i)}) \in L^\infty([0, T]; H^1(\mathbb{R}^n)) \times L^\infty([0, T]; L^2(\mathbb{R}^n))$ for $i = 1, 2$, then

$$\phi^{(1)} = \phi^{(2)}$$

almost everywhere in $[0, T] \times \mathbb{R}^n$.

The energy identity in Proposition 4.1 can also be localized in the following sense:

Proposition 4.4. *Define, for $0 \leq t \leq R$,*

$$E(t; x, R) := \frac{1}{2} \int_{B(x, R-t)} ((\partial_t\phi)^2 + \sum_{i=1}^n (\partial_i\phi)^2)(t, x) dx,$$

Then $E(t; x, R)$ is a monotonic decreasing function in t .

Proof. We start with $\partial_t \phi \square \phi = 0$ and integrate by parts in the region to the future of $\{t_1\} \times B(x, R)$ and to the past of $(\{t_2\} \times B(x, R - t_2)) \cup (\cup_{s \in [t_1, t_2]} (\{s\} \times \partial B(x, R - s)))$. The boundary terms on $\{t_1\} \times B(x, R - t_1)$ and $\{t_2\} \times B(x, R - t_2)$ give $E(t_1; x, R)$ and $E(t_2; x, R)$ respectively in a way similar to Proposition 4.1. The key point is that the boundary term on $\cup_{s \in [t_1, t_2]} (\{s\} \times \partial B(x, R - s))$ has a definite sign. More precisely,

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{B(x, R-t)} \left(\partial_t \phi (-\partial_t^2 \phi + \sum_{i=1}^n \partial_i^2 \phi) \right) (t, y) dy dt \\ &= -E(t_2; x, R) + E(t_1; x, R) - \frac{1}{2} \int_{t_1}^{t_2} \int_{\partial B(x, R-t)} \left((\partial_t \phi)^2 + \sum_{i=1}^n (\partial_i \phi)^2 - 2\partial_t \phi \sum_{i=1}^n \frac{(y-x)_i}{|y-x|} \partial_i \phi \right) (t, y) d\sigma dt. \end{aligned}$$

Here, $d\sigma$ is the surface measure on $\partial B(x, R - t)$. It remains to check that the last term is negative. To see this, notice that a simple application of Cauchy-Schwarz gives

$$\begin{aligned} 2\partial_t \phi \sum_{i=1}^n \frac{(y-x)_i}{|y-x|} \partial_i \phi &\leq (\partial_t \phi)^2 + \left(\sum_{i=1}^n \frac{(y-x)_i}{|y-x|} \partial_i \phi \right)^2 \\ &\leq (\partial_t \phi)^2 + \left(\sum_{i=1}^n \left(\frac{(y-x)_i}{|y-x|} \right)^2 \right) \left(\sum_{i=1}^n (\partial_i \phi)^2 \right) \leq (\partial_t \phi)^2 + \sum_{i=1}^n (\partial_i \phi)^2, \end{aligned}$$

which implies

$$(\partial_t \phi)^2 + \sum_{i=1}^n (\partial_i \phi)^2 - 2\partial_t \phi \sum_{i=1}^n \frac{(y-x)_i}{|y-x|} \partial_i \phi \geq 0.$$

□

An easy consequence of the localized energy estimates is the finite speed of propagation.

Corollary 4.5. *If $(\phi_0, \phi_1) = (0, 0)$ in $\{y \in \mathbb{R}^n : |y - x| \leq t\}$, then $\phi(t, x) = 0$.*

Notice that we have not yet proved the dispersion of the solutions to the linear wave equation using the energy method. This will be an important topic in later lectures. Instead, we now consider linear estimates for a more general class of (non-constant coefficient!) linear wave equation. We first define the class we consider:

$$(4.2) \quad \begin{cases} \partial_\alpha (a^{\alpha\beta} \partial_\beta \phi) + b^\alpha \partial_\alpha \phi = F \\ (\phi, \partial_t \phi) \upharpoonright_{\{t=0\}} = (\phi_0, \phi_1). \end{cases}$$

We require that a is a symmetric matrix and satisfy

$$(4.3) \quad \sum_{\alpha, \beta} |a^{\alpha\beta} - m^{\alpha\beta}| < \frac{1}{2}.$$

We will also need some regularity assumptions on a , b and F . We will state the precise conditions we need later on. For this class of equations, we do not in general have conservation of energy, but we can nonetheless show that some appropriately defined “energy” has the property that its growth is controlled. More precisely, we have

Theorem 4.6. *Let ϕ be a solution to (4.2), then for some constant $C = C(n) > 0$, the following energy estimates hold:*

$$\begin{aligned} & \sup_{t \in [0, T]} \|\partial \phi\|_{L^2(\mathbb{R}^n)}(t) \\ & \leq C \left(\|(\phi_0, \phi_1)\|_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} + \int_0^T \|F\|_{L^2(\mathbb{R}^n)}(t) dt \right) \exp\left(C \int_0^T (\|\partial a\|_{L^\infty(\mathbb{R}^n)}(t) + \|b\|_{L^\infty(\mathbb{R}^n)}(t)) dt\right). \end{aligned}$$

Proof. The proof is in fact the similar to that for the constant coefficient linear wave equation. We use the identity

$$(4.4) \quad \partial_t \phi (\partial_\alpha (a^{\alpha\beta} \partial_\beta \phi) + b^\alpha \partial_\alpha \phi - F) = 0$$

and integrate the first term by parts. We consider three different contributions. First, we look at the case when $\alpha = \beta = 0$:

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^n} \partial_t \phi \partial_t (a^{tt} \partial_t \phi) dx dt \\
(4.5) \quad &= \int_0^T \int_{\mathbb{R}^n} (\partial_t a^{tt}) (\partial_t \phi)^2 + \frac{1}{2} \partial_t (\partial_t \phi)^2 a^{tt} dx dt \\
&= \frac{1}{2} \int_{\{T\} \times \mathbb{R}^n} a^{tt} (\partial_t \phi)^2 dx - \frac{1}{2} \int_{\{0\} \times \mathbb{R}^n} a^{tt} (\partial_t \phi)^2 dx + \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} (\partial_t a^{tt}) (\partial_t \phi)^2 dx dt.
\end{aligned}$$

When we only have $i, j = 1, 2, \dots, n$, we have the following identity (note that we use the convention where $i, j = 1, 2, \dots, n$ and repeated indices are summed over):

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^n} \partial_t \phi \partial_i (a^{ij} \partial_j \phi) dx dt \\
(4.6) \quad &= -\frac{1}{2} \int_0^T \int_{\mathbb{R}^n} (\partial_t \partial_i \phi a^{ij} \partial_j \phi + \partial_t \partial_j \phi a^{ij} \partial_i \phi) dx dt \\
&= -\frac{1}{2} \int_0^T \int_{\mathbb{R}^n} \partial_t (\partial_i \phi \partial_j \phi) a^{ij} dx dt \\
&= -\frac{1}{2} \int_{\{T\} \times \mathbb{R}^n} a^{ij} \partial_i \phi \partial_j \phi dx - \frac{1}{2} \int_{\{0\} \times \mathbb{R}^n} a^{ij} \partial_i \phi \partial_j \phi dx + \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} (\partial_t a^{ij}) (\partial_i \phi \partial_j \phi) dx dt.
\end{aligned}$$

Notice that in the derivation above, we have used the symmetry of a .

Finally, for the term with t and $i = 1, \dots, n$, we have

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^n} \partial_t \phi (\partial_i (a^{i0} \partial_t \phi) + \partial_t (a^{i0} \partial_i \phi)) dx dt \\
(4.7) \quad &= \int_0^T \int_{\mathbb{R}^n} (\partial_i a^{i0}) (\partial_t \phi)^2 + a^{i0} \partial_i (\partial_t \phi)^2 + (\partial_t a^{i0}) (\partial_t \phi) (\partial_i \phi) dx dt \\
&= \int_0^T \int_{\mathbb{R}^n} (\partial_i a^{i0}) (\partial_t \phi)^2 - (\partial_i a^{i0}) (\partial_t \phi)^2 + (\partial_t a^{i0}) (\partial_t \phi) (\partial_i \phi) dx dt.
\end{aligned}$$

We now combine the equation (4.4) with the integrated identities (4.5), (4.6) and (4.7) to get

$$\begin{aligned}
(4.8) \quad & \left| \frac{1}{2} \int_{\{T\} \times \mathbb{R}^n} a^{tt} (\partial_t \phi)^2 dx - \frac{1}{2} \int_{\{T\} \times \mathbb{R}^n} a^{ij} \partial_i \phi \partial_j \phi dx \right| \\
& \leq \left| \frac{1}{2} \int_{\{0\} \times \mathbb{R}^n} a^{tt} (\partial_t \phi)^2 dx - \frac{1}{2} \int_{\{0\} \times \mathbb{R}^n} a^{ij} \partial_i \phi \partial_j \phi dx \right| \\
& \quad + C \int_0^T \|\partial \phi\|_{L^2(\mathbb{R}^n)} \|F\|_{L^2(\mathbb{R}^n)} dt \\
& \quad + C \int_0^T \left(\|\partial a\|_{L^\infty(\mathbb{R}^n)}(t) \|\partial \phi\|_{L^2(\mathbb{R}^n)}^2(t) + \|b\|_{L^\infty(\mathbb{R}^n)}(t) \|\partial \phi\|_{L^2(\mathbb{R}^n)}^2(t) \right) dt.
\end{aligned}$$

By the assumption (4.3), there exists a constant C such that the left hand side of (4.8) controls the L^2 norm of all derivatives of ϕ , i.e.,

$$\begin{aligned}
(4.9) \quad & \|\partial \phi\|_{L^2(\mathbb{R}^n)}(T) \\
& \leq C \left| \frac{1}{2} \int_{\{T\} \times \mathbb{R}^n} a^{tt} (\partial_t \phi)^2 dx - \frac{1}{2} \int_{\{T\} \times \mathbb{R}^n} a^{ij} \partial_i \phi \partial_j \phi dx \right|.
\end{aligned}$$

Combining (4.8) and (4.9), we get

$$\begin{aligned} & \|\partial\phi\|_{L^2(\mathbb{R}^n)}^2(T) \\ & \leq C\|\partial\phi\|_{L^2(\mathbb{R}^n)}^2(0) + \int_0^T \|\partial\phi\|_{L^2(\mathbb{R}^n)}\|F\|_{L^2(\mathbb{R}^n)}dt \\ & \quad + C \int_0^T \left(\|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\|\partial\phi\|_{L^2(\mathbb{R}^n)}^2(t) + \|a\|_{L^\infty(\mathbb{R}^n)}(t)\|\partial\phi\|_{L^2(\mathbb{R}^n)}^2(t) \right) dt. \end{aligned}$$

Now, notice that we can in fact have a stronger estimate on the left hand side which control the supremum of the $\cdot H^1$ norm over the interval $[0, T]$. This is because we can perform the argument above in a smaller interval of time. Hence,

$$\begin{aligned} & \sup_{t \in [0, T]} \|\partial\phi\|_{L^2(\mathbb{R}^n)}^2(t) \\ & \leq C\|\partial\phi\|_{L^2(\mathbb{R}^n)}^2(0) + \int_0^T \|\partial\phi\|_{L^2(\mathbb{R}^n)}\|F\|_{L^2(\mathbb{R}^n)}dt \\ & \quad + C \int_0^T \left(\|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\|\partial\phi\|_{L^2(\mathbb{R}^n)}^2(t) + \|a\|_{L^\infty(\mathbb{R}^n)}(t)\|\partial\phi\|_{L^2(\mathbb{R}^n)}^2(t) \right) dt. \end{aligned}$$

Now, we can use Cauchy-Schwarz to get

$$\begin{aligned} & \int_0^T \|\partial\phi\|_{L^2(\mathbb{R}^n)}\|F\|_{L^2(\mathbb{R}^n)}dt \\ & \leq \delta \sup_{t \in [0, T]} \|\partial\phi\|_{L^2(\mathbb{R}^n)}^2(t) + C\delta^{-1} \left(\int_0^T \|F\|_{L^2(\mathbb{R}^n)}^2 dt \right). \end{aligned}$$

Choosing $\delta > 0$ sufficiently small, we can therefore absorb the term $\delta \sup_{t \in [0, T]} \|\partial\phi\|_{L^2(\mathbb{R}^n)}^2(t)$ to the left hand side to get

$$\begin{aligned} & \sup_{t \in [0, T]} \|\partial\phi\|_{L^2(\mathbb{R}^n)}^2(t) \\ & \leq C\|\partial\phi\|_{L^2(\mathbb{R}^n)}^2(0) + \left(\int_0^T \|F\|_{L^2(\mathbb{R}^n)}^2 dt \right) \\ & \quad + C \int_0^T \left(\|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\|\partial\phi\|_{L^2(\mathbb{R}^n)}^2(t) + \|a\|_{L^\infty(\mathbb{R}^n)}(t)\|\partial\phi\|_{L^2(\mathbb{R}^n)}^2(t) \right) dt. \end{aligned}$$

The conclusion follows from Gronwall's lemma below. □

We recall here the Gronwall's lemma:

Lemma 4.7. *Let $f(t)$ be a positive continuous function and $g(t)$ be a positive integrable function such that*

$$f(t) \leq A + \int_0^t f(s)g(s)ds$$

for some $A \geq 0$ for every $t \in [0, T]$. Then

$$f(t) \leq A \exp\left(\int_0^t g(s)ds\right)$$

for every $t \in [0, T]$.

Proof. We will actually give two proofs. The second proof gives a slightly weaker result that is stated above, but we use this to illustrate a method that is known as the **bootstrap** method. This method will be important later when we consider nonlinear problems.

(1) Differentiating and using the given inequality, we have

$$\frac{d}{dt} \left(A + \int_0^t f(s)g(s)ds \right) = f(t)g(t) \leq g(t) \left(A + \int_0^t f(s)g(s)ds \right).$$

Integrating, we obtain

$$\frac{A + \int_0^t f(s)g(s)ds}{A} \leq \exp\left(\int_0^t g(s)ds\right).$$

Use the given inequality again to get the desired conclusion.

(2) For the second proof, we will only obtain the weaker conclusion

$$(4.10) \quad f(t) \leq 2A \exp\left(4 \int_0^t g(s)ds\right).$$

Define the subset of $[0, T]$ by $B := \{t \in [0, T] : (4.10) \text{ holds for every } s \in [0, t]\}$. We will show that B is non-empty, open and close - thus $B = [0, T]$.

- B is obviously non-empty since $0 \in B$.
- B is obviously closed by the continuity of f .
- The only difficult part is to show that B is open. By the continuity of f , it suffices to show that if $t \in B$, then we have

$$f(t) \leq \frac{3A}{2} \exp\left(4 \int_0^t g(s)ds\right),$$

i.e., a bound that improves over (4.10). To show this, observe that if $t \in B$, then

$$\begin{aligned} f(t) &\leq A + \int_0^t f(s)g(s)ds \leq A + 2A \int_0^t g(s) \exp\left(4 \int_0^s g(r)dr\right)ds \\ &\leq A\left(1 + \frac{1}{2}(\exp\left(4 \int_0^t g(s)ds\right) - 1)\right) \leq \frac{3A}{2} \exp\left(4 \int_0^t g(s)ds\right). \end{aligned}$$

□

Using Gronwall's lemma, we have thus concluded the proof of the energy estimates (Theorem 4.6). As in the case for the constant coefficient linear wave equation, an immediate consequence of the energy estimates is the uniqueness of solutions:

Corollary 4.8. *The conclusion of Corollary 4.3 also holds for the more general class of (non-constant coefficient) linear wave equation that is being considered!*

For the constant coefficient linear wave equation (4.1), we know that if ϕ is a solution, that $\partial_t \phi$ and $\partial_{x_i} \phi$ are also solutions. As a consequence, say if the initial data are smooth and compactly supported, all higher derivatives for ϕ are bounded in L^2 . In general, of course ϕ being a solution does not imply that its derivatives are also solutions. However as a consequence of Theorem 4.6, we can also control higher derivatives of ϕ in L^2 . More precisely, we have the following corollary:

Corollary 4.9. *Let ϕ be a solution to (4.2), then for some constant $C = C(n, k) > 0$, the following energy estimates hold:*

$$\begin{aligned} &\sup_{t \in [0, T]} \sum_{1 \leq |\alpha| \leq k} \|\partial^\alpha \phi\|_{L^2(\mathbb{R}^n)}(t) \\ &\leq C \left(\|(\phi_0, \phi_1)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)} + \int_0^T \|F\|_{H^{k-1}(\mathbb{R}^n)}(t)dt \right. \\ &\quad \left. + \int_0^T \left(\sum_{\substack{|\alpha|+|\beta| \leq k+1 \\ 1 \leq |\alpha|, |\beta| \leq k}} \|\partial^\alpha a \partial^\beta \phi\|_{L^2(\mathbb{R}^n)}(t) + \sum_{\substack{|\alpha|+|\beta| \leq k \\ 1 \leq |\beta| \leq k}} \|\partial^\alpha b \partial^\beta \phi\|_{L^2(\mathbb{R}^n)}(t) \right) dt \right) \\ &\quad \times \exp \left(C \int_0^T (\|\partial a\|_{L^\infty(\mathbb{R}^n)}(t) + \|b\|_{L^\infty(\mathbb{R}^n)}(t))dt \right). \end{aligned}$$

Proof. To see this take a multi-index α' such that $0 \leq |\alpha'| \leq k-1$. Derive a wave equation for $\partial^{\alpha'} \phi$ and apply Theorem 4.6. □

It is easy to see that we can in fact also control the L^2 norm itself:

Corollary 4.10. *Let ϕ be a solution to (4.2) and k be a positive integer. Then for some constant $C = C(n, k) > 0$, the following energy estimates hold:*

$$\begin{aligned} & \sup_{t \in [0, T]} \|(\phi, \partial_t \phi)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)}(t) \\ & \leq C(1 + T) \left(\|(\phi_0, \phi_1)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)} + \int_0^T \|F\|_{H^{k-1}(\mathbb{R}^n)}(t) dt \right. \\ & \quad \left. + \int_0^T \left(\sum_{\substack{|\alpha|+|\beta| \leq k+1 \\ 1 \leq |\alpha|, |\beta| \leq k}} \|\partial^\alpha a \partial^\beta \phi\|_{L^2(\mathbb{R}^n)}(t) + \sum_{\substack{|\alpha|+|\beta| \leq k \\ 1 \leq |\beta| \leq k}} \|\partial^\alpha b \partial^\beta \phi\|_{L^2(\mathbb{R}^n)}(t) \right) dt \right) \\ & \quad \times \exp \left(C \int_0^T (\|\partial a\|_{L^\infty(\mathbb{R}^n)}(t) + \|b\|_{L^\infty(\mathbb{R}^n)}(t)) dt \right). \end{aligned}$$

Proof. The only term that has not already been controlled in Corollary 4.9 is

$$\sup_{t \in [0, T]} \|\phi\|_{L^2(\mathbb{R}^n)}(t).$$

On the other hand, it is easy to see (**Exercise**) that

$$\sup_{t \in [0, T]} \|\phi\|_{L^2(\mathbb{R}^n)}(t) \leq C \left(\|\phi_0\|_{L^2(\mathbb{R}^n)} + \int_0^T \|\partial_t \phi\|_{L^2(\mathbb{R}^n)}(t) dt \right).$$

We can then apply Corollary 4.9 to control the right hand side to obtain the desired conclusion. \square

5. EXISTENCE OF SOLUTIONS TO GENERAL (NON-CONSTANT COEFFICIENT) LINEAR WAVE EQUATION

In the previous lecture, we showed that if a solution exists, then it must obey certain energy estimates. In particular, the existence of solution is assumed. Now, we show that in fact using the energy estimates together with some abstract theory, we can obtain existence of solutions. For technical reasons, it is convenient to look at a smaller subclass than the previous section. Consider the equation

$$(5.1) \quad \begin{cases} \partial_\alpha (a^{\alpha\beta} \partial_\beta \phi) + b^\alpha \partial_\alpha \phi = F \\ (\phi, \partial_t \phi) \upharpoonright_{\{t=0\}} = (\phi_0, \phi_1) \in H^k \times H^{k-1}. \end{cases}$$

We require

$$|a^{\alpha\beta} - m^{\alpha\beta}| < \frac{1}{2}, \quad \partial_\alpha b^\alpha = 0.$$

To simplify the analysis below, we assume that a , b and all the derivatives (of all orders) are bounded in $[0, T] \times \mathbb{R}^n$.

Before we state and prove the existence result, we need the following lemma:

Lemma 5.1. *Let*

$$L^* \psi := \partial_\alpha (a^{\alpha\beta} \partial_\beta \psi) - \partial_\alpha (b^\alpha \psi),$$

be defined as the (formal) adjoint of L , defined as

$$L^* \phi := \partial_\alpha (a^{\alpha\beta} \partial_\beta \phi) + b^\alpha \partial_\alpha \phi.$$

Suppose $\psi \in C_c^\infty((-\infty, T) \times \mathbb{R}^n)$, then for every $m \in \mathbb{Z}$, there exists $C = C(m, T, a, b) > 0$ such that

$$\|\psi\|_{H^m(\mathbb{R}^n)}(t) \leq C \int_t^T \|L^* \psi\|_{H^{m-1}(\mathbb{R}^n)}(s) ds$$

for every $t \in [0, T]$.

Proof. For $m \geq 1$, this is simply a consequence of the energy estimates³. We now carry out an induction for the cases $m \leq 0$. Assume that the result holds for some $m_0 + 2$, we wish to prove the same result (with a possibly different constant) for m_0 . To this end, consider

$$\Psi = (1 - \Delta)^{-1}\psi,$$

defined, for instance, using the Fourier transform. Then, there exist $C > 0$ depending on m_0 , T , a and b such that

$$|L^*\psi - (1 - \Delta)L^*\Psi| \leq C \left(\sum_{1 \leq |\alpha| \leq 3} |\partial^\alpha \phi| \right).$$

Therefore,

$$\|L^*\Psi\|_{H^{m_0+1}(\mathbb{R}^3)}(t) \leq C \left(\|L^*\psi\|_{H^{m_0-1}(\mathbb{R}^3)}(t) + \|\Psi\|_{H^{m_0+2}(\mathbb{R}^3)}(t) \right).$$

Therefore, by the induction hypothesis,

$$\begin{aligned} & \|\Psi\|_{H^{m_0+2}(\mathbb{R}^3)}(t) \\ & \leq C \int_t^T \left(\|L^*\psi\|_{H^{m_0-1}(\mathbb{R}^3)}(s) + \|\Psi\|_{H^{m_0+2}(\mathbb{R}^3)}(s) \right) ds \\ & \leq C \int_t^T \|L^*\psi\|_{H^{m_0-1}(\mathbb{R}^3)}(s) ds, \end{aligned}$$

where in the last line we have used Gronwall's lemma, noting that the constant is allowed to depend on T . This implies

$$\|\psi\|_{H^{m_0}(\mathbb{R}^3)}(t) \leq C \|\Psi\|_{H^{m_0+2}(\mathbb{R}^3)}(t) \leq C \int_t^T \|L^*\psi\|_{H^{m_0-1}(\mathbb{R}^3)}(s) ds.$$

□

We are now ready for the main result in this lecture:

Theorem 5.2 (Existence of solutions to the linear equation). *Given $F \in L^1([0, T]; H^{k-1}(\mathbb{R}^n))$, there exists a solution*

$$(\phi, \partial_t \phi) \in L^\infty([0, T]; H^k(\mathbb{R}^n)) \times L^\infty([0, T]; H^{k-1}(\mathbb{R}^n))$$

solving (5.1).

Proof. We begin with the case where $(\phi_0, \phi_1) = (0, 0)$. For every element in $L^*(C_c^\infty((-\infty, T) \times \mathbb{R}^n))$, define a map to \mathbb{R} by

$$L^*\psi \mapsto \int_0^T \int_{\mathbb{R}^n} \psi F dx dt =: \langle F, \psi \rangle.$$

(Note that this is well-defined because of uniqueness for $L^*\psi = f$ with zero data at $t = T$, a fact that we proved using energy estimates.) Notice that by the assumption on F and Lemma 5.1, we have the bound

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}^n} \psi F dx dt \right| & \leq C \left(\int_0^T \|F\|_{H^{k-1}(\mathbb{R}^n)}(t) dt \right) \left(\sup_{t \in [0, T]} \|\psi\|_{H^{-k+1}(\mathbb{R}^n)} \right) \\ & \leq C \int_0^T \|L^*\psi\|_{H^{-k}(\mathbb{R}^n)}(s) ds. \end{aligned}$$

Using Hahn-Banach, there exists a function $\phi \in (L^1((-\infty, T); H^{-k}(\mathbb{R}^n)))^* = L^\infty((-\infty, T); H^k(\mathbb{R}^n))$ with $\phi = 0$ for $t < 0$ that extends the map above, i.e.,

$$\langle F, \psi \rangle = \langle \phi, L^*\psi \rangle$$

for every $\psi \in C_c^\infty((-\infty, T) \times \mathbb{R}^n)$. Therefore, ϕ is a solution in the sense of distribution. Finally, we use the equation to show that $\phi \in C^1([0, T]; L^2)$ and therefore $(\phi, \partial_t \phi) \upharpoonright_{\{t=0\}} = (0, 0)$. This concludes the proof in the special case where $(\phi_0, \phi_1) = (0, 0)$.

Turning to the general case. Let u be a function in $[0, T] \times \mathbb{R}^n$ such that

$$(u, \partial_t u) \upharpoonright_{\{t=0\}} = (\phi_0, \phi_1).$$

³Notice that it is at this point that we use the condition $\partial_\alpha b^\alpha = 0$, which guarantees that L^* does not have zeroth order terms.

At the same time, solve

$$L\eta = F - Lu$$

with initial data

$$(\eta, \partial_t \eta) \upharpoonright_{\{t=0\}} = (0, 0).$$

Then letting $\phi := \eta + u$ gives the desired solution. \square

Remark 5.3. Notice that the above theorem gives only the existence of solutions in the sense of distribution. Nevertheless, using the Sobolev embedding theorem, if we assume that the initial data is sufficiently regular, then in fact the solution can be understood classically.

We recall here the following standard Sobolev embedding theorem which we referred to in the remark above. It will also make a few more appearances later.

Theorem 5.4 (Sobolev embedding theorem). *For every $s > \frac{n}{2}$, there exists $C = C(n, s) > 0$ such that*

$$\|\phi\|_{L^\infty(\mathbb{R}^n)} \leq C\|\phi\|_{H^s(\mathbb{R}^n)}.$$

6. LOCAL THEORY FOR NONLINEAR WAVE EQUATIONS

We consider quasilinear equations of the form

$$(6.1) \quad \begin{cases} \partial_\alpha(a^{\alpha\beta}(\phi)\partial_\beta\phi) = F(\phi, \partial\phi) \\ (\phi, \partial_t\phi) \upharpoonright_{\{t=0\}} = (\phi_0, \phi_1) \in H^k \times H^{k-1}. \end{cases}$$

We require that

$$(6.2) \quad |a^{\alpha\beta} - m^{\alpha\beta}| \leq 2$$

and

$$(6.3) \quad a^{\alpha\beta}, F \text{ are smooth functions of their arguments.}$$

As a consequence of (6.3), we have

$$(6.4) \quad \sum_{\alpha, \beta} \sum_{|x| \leq A} \sup_{|\gamma| \leq N} |\partial_x^\gamma(a^{\alpha\beta})|(x) \leq C_{A, N}$$

and

$$(6.5) \quad \sum_{|x|, \|p\| \leq A} \sup_{|\gamma| \leq N} |\partial_{x,p}^\gamma F|(x, p) \leq C_{A, N}.$$

We now state the main theorem of this section, which is a local existence and uniqueness theorem for this general class of equations.

Theorem 6.1. *Fix a and F satisfying the assumptions above and fix an integer $k \geq n + 2$.*

(1) *(Existence and uniqueness of local-in-time solutions) There exists⁴*

$$T = T(\|\phi_0\|_{H^k(\mathbb{R}^n)}, \|\phi_1\|_{H^{k-1}(\mathbb{R}^n)}) > 0$$

such that there exists a (classical) solution ϕ to (6.1) with

$$(\phi, \partial_t\phi) \in L^\infty([0, T]; H^k(\mathbb{R}^n)) \times L^\infty([0, T]; H^{k-1}(\mathbb{R}^n)).$$

Moreover, the solution is unique in the function space

$$(\phi, \partial_t\phi) \in L^\infty([0, T]; H^k(\mathbb{R}^n)) \times L^\infty([0, T]; H^{k-1}(\mathbb{R}^n)).$$

(2) *(Continuous dependence on initial data) Let $\phi_0^{(i)}, \phi_1^{(i)}$ be sequences of functions such that $\phi_0^{(i)} \rightarrow \phi_0$ in $H^k(\mathbb{R}^n)$ and $\phi_1^{(i)} \rightarrow \phi_1$ in $H^{k-1}(\mathbb{R}^n)$ as $i \rightarrow \infty$. Then taking $T > 0$ sufficiently small, we have*

$$\|(\phi^{(i)} - \phi, \partial_t(\phi^{(i)} - \phi))\|_{L^\infty([0, T]; H^s(\mathbb{R}^n)) \times L^\infty([0, T]; H^{s-1}(\mathbb{R}^n))} \rightarrow 0$$

as $i \rightarrow \infty$ for every $1 \leq s < k$. (Here, ϕ is the solution arising from data (ϕ_0, ϕ_1) and $\phi^{(i)}$ is the solution arising from data $(\phi_0^{(i)}, \phi_1^{(i)})$.)

⁴Of course T also depends on a and F but we will consider them fixed for this theorem.

(3) (*Breakdown criterion*) By part 1, there exists a maximal time of existence $T_* := \sup\{T > 0 : \text{conclusion of part 1 holds}\} > 0$. If $T_* < \infty$, then we have

$$\limsup_{t \rightarrow T_*} \sum_{|\alpha| \leq \lfloor \frac{n}{2} + 1 \rfloor} \|\partial_x^\alpha \phi\|_{L^\infty(\mathbb{R}^n)}(t) \rightarrow \infty.$$

(4) (*Persistence of regularity*) If the initial data are moreover in $H^m \times H^{m-1}$ for some $m > k$, then the solution $(\phi, \partial_t \phi) \in L^\infty([0, T]; H^m(\mathbb{R}^n)) \times L^\infty([0, T]; H^{m-1}(\mathbb{R}^n))$ for every $T < T_*$. In addition, if the initial data are smooth, then the solution is smooth in $[0, T_*) \times \mathbb{R}^n$.

Remark 6.2. An evolution equation is said to be well-posed in the sense of Hadamaard if there is existence, uniqueness of solutions and continuous dependence on initial data.

We now turn to the proof of the theorem:

Proof. (1) This part of the theorem is proved by Picard's iteration. By a density argument, it suffices to assume that $(\phi_0, \phi_1) \in \mathcal{S} \times \mathcal{S}$. Define a sequence of (smooth) functions $\phi^{(i)}$ for $i \geq 1$ such that

$$\phi^{(1)} = 0$$

and $\phi^{(i)}$ is defined iteratively for $i \geq 2$ as the unique solution to

$$(6.6) \quad \begin{cases} \partial_\alpha (a^{\alpha\beta}(\phi^{(i-1)}) \partial_\beta \phi^{(i)}) = F(\phi^{(i-1)}, \partial \phi^{(i-1)}) \\ (\phi^{(i)}, \partial_t \phi^{(i)}) \upharpoonright_{\{t=0\}} = (\phi_0, \phi_1). \end{cases}$$

We will show two properties of the sequence $\phi^{(i)}$. First, for $T > 0$ sufficiently small (depending only on $\|\phi_0\|_{H^k(\mathbb{R}^n)}$ and $\|\phi_1\|_{H^{k-1}(\mathbb{R}^n)}$), the sequence is uniformly (in i) bounded in $L^\infty([0, T]; H^k(\mathbb{R}^n)) \times L^\infty([0, T]; H^{k-1}(\mathbb{R}^n))$, i.e.,

$$(6.7) \quad \|(\phi^{(i)}, \partial_t \phi^{(i)})\|_{L^\infty([0, T]; H^k(\mathbb{R}^n)) \times L^\infty([0, T]; H^{k-1}(\mathbb{R}^n))} \leq C$$

for some $C > 0$ (depending on k, n, F and a , but independent on i). Then, we show that for $T > 0$ chosen to be smaller if necessary, $(\phi^{(i)}, \partial_t \phi^{(i)})$ is a Cauchy sequence in $L^\infty([0, T]; H^1(\mathbb{R}^n)) \times L^\infty([0, T]; L^2(\mathbb{R}^n))$.

(a) We begin with the first part, the idea is to use the energy estimates. Obviously, it suffices to prove that there exists some $A > 0$ such that

$$\begin{aligned} & \|(\phi^{(i-1)}, \partial_t \phi^{(i-1)})\|_{L^\infty([0, T]; H^k(\mathbb{R}^n)) \times L^\infty([0, T]; H^{k-1}(\mathbb{R}^n))} \leq A \\ \implies & \|(\phi^{(i)}, \partial_t \phi^{(i)})\|_{L^\infty([0, T]; H^k(\mathbb{R}^n)) \times L^\infty([0, T]; H^{k-1}(\mathbb{R}^n))} \leq A. \end{aligned}$$

To this end, we assume

$$(6.8) \quad \|(\phi^{(i-1)}, \partial_t \phi^{(i-1)})\|_{L^\infty([0, T]; H^k(\mathbb{R}^n)) \times L^\infty([0, T]; H^{k-1}(\mathbb{R}^n))} \leq A$$

for some $A > 0$ to be chosen below. In fact, it is convenient to use the bootstrap method and assume also that

$$(6.9) \quad \|(\phi^{(i)}, \partial_t \phi^{(i)})\|_{L^\infty([0, T]; H^k(\mathbb{R}^n)) \times L^\infty([0, T]; H^{k-1}(\mathbb{R}^n))} \leq 4A.$$

Of course, we can make this assumption as long as at the end we can improve the constant in (6.9).

Recall that in order to apply the energy estimates in Corollary 4.10, we need a bound on

$$\int_0^T \sum_{0 \leq |\alpha| \leq k-1} \|\partial^\alpha F(\phi^{(i-1)}, \partial \phi^{(i-1)})\|_{L^2(t)} dt$$

and

$$\int_0^T \sum_{\substack{|\gamma|+|\sigma| \leq k+1 \\ 1 \leq |\gamma|, |\sigma| \leq k}} \sum_{\alpha, \beta} \|\partial^\gamma (a^{\alpha\beta}(\phi^{(i-1)})) \partial^\sigma \phi^{(i)}\|_{L^2(t)} dt.$$

To obtain such bounds, we need to use the smoothness of F and a as well as the Sobolev embedding theorem (Theorem 5.4). First, for F , we notice that by the chain rule, Theorem 5.4 and (6.5), there exists $B = B(A, n, k, F) > 0$ such that for $t \in [0, T]$, we have

$$\begin{aligned} & \sum_{0 \leq |\alpha| \leq k-1} \|\partial^\alpha F(\phi^{(i-1)}, \partial \phi^{(i-1)})\|_{L^2}(t) \\ & \leq C \left(\sum_{0 \leq |\alpha| \leq k} \|\partial^\alpha \phi^{(i-1)}\|_{L^2}(t) + \sum_{0 \leq |\alpha_1| + |\alpha_2| \leq k-1} \|\partial \partial^{\alpha_1} \phi^{(i-1)} \partial \partial^{\alpha_2} \phi^{(i-1)}\|_{L^2}(t) \right. \\ & \quad \left. + \text{cubic terms} + \dots + \text{terms of order } k-1 \right) \end{aligned}$$

Consider for instance the quadratic term

$$\sum_{0 \leq |\alpha_1| + |\alpha_2| \leq k-1} \|\partial \partial^{\alpha_1} \phi^{(i-1)} \partial \partial^{\alpha_2} \phi^{(i-1)}\|_{L^2(\mathbb{R}^n)}(t).$$

Notice now that either $|\alpha_1| \leq \frac{k-1}{2}$ or $|\alpha_2| \leq \frac{k-1}{2}$. So we can assume without loss of generality that $|\alpha_1| \leq \frac{k-1}{2}$. On the other hand, since $(\frac{k-1}{2} + 1) + \frac{n}{2} < k$ (by the assumption $k \geq n+2$), we can apply Theorem 5.4 to get

$$\sum_{0 \leq |\alpha_1| \leq \frac{k-1}{2}} \|\partial \partial^{\alpha_1} \phi^{(i-1)}\|_{L^\infty(\mathbb{R}^n)} \leq C \sum_{0 \leq |\alpha| \leq k} \|\partial^\alpha \phi^{(i-1)}\|_{L^2}(t).$$

Therefore,

$$\sum_{0 \leq |\alpha_1| + |\alpha_2| \leq k-1} \|\partial \partial^{\alpha_1} \phi^{(i-1)} \partial \partial^{\alpha_2} \phi^{(i-1)}\|_{L^2(\mathbb{R}^n)}(t) \leq C \sum_{0 \leq |\alpha| \leq k} \|\partial^\alpha \phi^{(i-1)}\|_{L^2(\mathbb{R}^n)}(t)^2.$$

In a similar manner, we can treat all the higher order terms to get

$$\begin{aligned} & \sum_{0 \leq |\alpha| \leq k-1} \|\partial^\alpha F(\phi^{(i-1)}, \partial \phi^{(i-1)})\|_{L^2}(t) \\ & \leq C \left(1 + \sum_{0 \leq |\alpha| \leq k} \|\partial^\alpha \phi^{(i-1)}\|_{L^2(\mathbb{R}^n)}(t) \right)^k \leq B, \end{aligned}$$

where in the last line we have used (6.8).

In a similar fashion, we can also obtain

$$\sum_{\substack{|\gamma| + |\sigma| \leq k+1 \\ 1 \leq |\gamma|, |\sigma| \leq k}} \sum_{\alpha, \beta} \|\partial^\gamma (a^{\alpha\beta}(\phi^{(i-1)})) \partial^\sigma \phi^{(i)}\|_{L^2}(t) \leq B$$

for some $B = B(A, n, k, F) > 0$, where we have used the bootstrap assumption (6.9) in addition to the induction hypothesis (6.8). Moreover, we have a bound

$$\sum_{|\gamma|=1} \|\partial^\gamma (a^{\alpha\beta}(\phi^{(i-1)}))\|_{L^\infty(\mathbb{R}^n)} \leq B$$

using Proposition 5.4 and (6.8).

We now apply the energy estimates in Corollary 4.10 to obtain

$$\begin{aligned} & \|(\phi^{(i)}, \partial_t \phi^{(i)})\|_{L^\infty([0, T]; H^k(\mathbb{R}^n)) \times L^\infty([0, T]; H^{k-1}(\mathbb{R}^n))} \\ & \leq C (\|(\phi_0, \phi_1)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)} + BT + CBT^2) \times \exp(CBT). \end{aligned}$$

In order to fix notations, let's call the constant in the above inequality C_0 and assume without loss of generality that $C_0 \geq 2$. Here is the key point: while B can be very large and depends on A , we can choose $T > 0$ to be sufficiently small (depending on A) such that

$$(BT + C_0 BT^2) \exp(C_0 BT) \leq \|(\phi_0, \phi_1)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)}$$

and

$$\exp(C_0 BT) \leq 2.$$

With T chosen as above, we thus have

$$\|(\phi^{(i)}, \partial_t \phi^{(i)})\|_{L^\infty([0, T]; H^k(\mathbb{R}^n)) \times L^\infty([0, T]; H^{k-1}(\mathbb{R}^n))} \leq 4C_0 \|(\phi_0, \phi_1)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)}.$$

Now, let's choose $A = 4C_0 \|(\phi_0, \phi_1)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)}$. We have thus shown the desired implication

$$\begin{aligned} & \|(\phi^{(i-1)}, \partial_t \phi^{(i-1)})\|_{L^\infty([0, T]; H^k(\mathbb{R}^n)) \times L^\infty([0, T]; H^{k-1}(\mathbb{R}^n))} \leq A \\ \implies & \|(\phi^{(i)}, \partial_t \phi^{(i)})\|_{L^\infty([0, T]; H^k(\mathbb{R}^n)) \times L^\infty([0, T]; H^{k-1}(\mathbb{R}^n))} \leq A. \end{aligned}$$

for this A , provided that $T > 0$ is sufficiently small depending on A . In particular, we have improved the constant in the bootstrap assumption (6.9). On the other hand, A depends only on $\|(\phi_0, \phi_1)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)}$ and therefore T indeed depends only on $\|(\phi_0, \phi_1)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)}$, as desired.

We have thus finished the first part of the proof.

- (b) We now move to the second part in which we show that the sequence is Cauchy in a larger space $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. To this end, for every $i \geq 3$, we consider the equation for $\phi^{(i)} - \phi^{(i-1)}$:

$$(6.10) \quad \begin{aligned} & \partial_\alpha (a^{\alpha\beta}(\phi^{(i-1)}) \partial_\beta (\phi^{(i)} - \phi^{(i-1)})) \\ &= -\partial_\alpha ((a^{\alpha\beta}(\phi^{(i-1)}) - a^{\alpha\beta}(\phi^{(i-2)})) \partial_\beta \phi^{(i-1)}) + F(\phi^{(i-1)}, \partial \phi^{(i-1)}) - F(\phi^{(i-2)}, \partial \phi^{(i-2)}). \end{aligned}$$

Since a is continuously differentiable, we can use (6.4) together with the bound (6.7) obtained in part (a), we have for some $C > 0$ (depending on initial data but *independent* of i and T) that

$$\left| \partial_\alpha ((a^{\alpha\beta}(\phi^{(i-1)}) - a^{\alpha\beta}(\phi^{(i-2)})) \partial_\beta \phi^{(i-1)}) \right| \leq C \left| \partial(\phi^{(i-1)} - \phi^{(i-2)}) \right|$$

which implies using (6.7) again that

$$\|\partial_\alpha ((a^{\alpha\beta}(\phi^{(i-1)}) - a^{\alpha\beta}(\phi^{(i-2)})) \partial_\beta \phi^{(i-1)})\|_{L^2(\mathbb{R}^n)}(t) \leq C \|\partial(\phi^{(i-1)} - \phi^{(i-2)})\|_{L^2(\mathbb{R}^n)}(t)$$

for every $t \in [0, T]$. Similarly, using (6.5) and (6.7), we have

$$\|F(\phi^{(i-1)}, \partial \phi^{(i-1)}) - F(\phi^{(i-2)}, \partial \phi^{(i-2)})\|_{L^2(\mathbb{R}^n)}(t) \leq C \|\phi^{(i-1)} - \phi^{(i-2)}\|_{H^1(\mathbb{R}^n)}(t)$$

for some $C = C(\|(\phi_0, \phi_1)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)}) > 0$. It is important that C is *independent* of i . Also, using the bound in part (a), we have,

$$\|\partial^\gamma (a^{\alpha\beta}(\phi^{(i-1)}))\|_{L^\infty(\mathbb{R}^n)}(t) \leq C.$$

Therefore, applying Corollary 4.10 to (6.10), (noticing that the data for $\phi^{(i)}$ and $\phi^{(i-1)}$ coincide if $i \geq 2$), we get

$$\begin{aligned} & \sup_{t \in [0, T]} \|(\phi^{(i)} - \phi^{(i-1)}, \partial_t \phi^{(i)} - \partial_t \phi^{(i-1)})\|_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}(t) \\ & \leq CT \sup_{t \in [0, T]} \|\phi^{(i-1)} - \phi^{(i-2)}\|_{H^1(\mathbb{R}^n)}(t). \end{aligned}$$

Here, C can depend on T only in a non-decreasing manner. By (6.7),

$$\sup_{t \in [0, T]} \|\phi^{(2)} - \phi^{(1)}\|_{H^1(\mathbb{R}^n)}(t) \leq C_1.$$

Therefore, choosing T to be sufficiently small, we have for $i \geq 3$,

$$\sup_{t \in [0, T]} \|\phi^{(i)} - \phi^{(i-1)}\|_{H^1(\mathbb{R}^n)}(t) \leq \frac{1}{2} \sup_{t \in [0, T]} \|\phi^{(i-1)} - \phi^{(i-2)}\|_{H^1(\mathbb{R}^n)}(t),$$

which implies

$$\sup_{t \in [0, T]} \|\phi^{(i)} - \phi^{(i-1)}\|_{H^1(\mathbb{R}^n)}(t) \leq 2^{-i+2} C_1.$$

From this the Cauchy property follows straightforwardly.

This concludes the proof of existence. The proof of uniqueness can be carried out easily by considering the equation satisfied by the difference of two solutions. However, we will not carry out this in detail since uniqueness can alternatively be derived from the continuous dependence of initial data that we will prove immediately below.

- (2) Pick some i and we bound the difference $\phi^{(i)} - \phi$ using the energy estimates. Writing the equation for the difference

$$\begin{aligned} & \partial_\alpha (a^{\alpha\beta}(\phi^{(i)}) \partial_\beta (\phi^{(i)} - \phi)) \\ &= -\partial_\alpha ((a^{\alpha\beta}(\phi^{(i)}) - a^{\alpha\beta}(\phi)) \partial_\beta \phi) + F(\phi^{(i)}, \partial\phi^{(i)}) - F(\phi, \partial\phi). \end{aligned}$$

Applying the energy estimates, we get that the following holds for some $C > 0$ independent of i and for all $t \in [0, T]$

$$\begin{aligned} & \sup_{s \in [0, t]} \|(\phi^{(i)} - \phi, \partial_t \phi^{(i)} - \partial_t \phi)\|_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}(s) \\ & \leq C \left(\|(\phi_0^{(i)} - \phi_0, \phi_1^{(i)} - \phi_1)\|_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} + \int_0^t \|\partial_\alpha ((a^{\alpha\beta}(\phi^{(i)}) - a^{\alpha\beta}(\phi)) \partial_\beta \phi)\|_{L^2(\mathbb{R}^n)}(t) dt \right. \\ & \quad \left. + \int_0^t \|F(\phi^{(i)}, \partial\phi^{(i)}) - F(\phi, \partial\phi)\|_{L^2(\mathbb{R}^n)}(t) dt \right). \end{aligned}$$

Now, applying the bounds for $\phi^{(i)}$ and ϕ that we obtained in part (1) of the theorem, we get

$$\begin{aligned} & \sup_{s \in [0, t]} \|(\phi^{(i)} - \phi, \partial_t \phi^{(i)} - \partial_t \phi)\|_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}(s) \\ & \leq C \left(\|(\phi_0^{(i)} - \phi_0, \phi_1^{(i)} - \phi_1)\|_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} + \int_0^t \|\phi^{(i)} - \phi\|_{H^1(\mathbb{R}^n)}(t) dt \right). \end{aligned}$$

By Gronwall's inequality, we thus have that for some $C = C(T) > 0$,

$$\sup_{t \in [0, T]} \|(\phi^{(i)} - \phi, \partial_t \phi^{(i)} - \partial_t \phi)\|_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}(t) \leq C \|(\phi_0^{(i)} - \phi_0, \phi_1^{(i)} - \phi_1)\|_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}.$$

Since the right hand side $\rightarrow 0$ as $i \rightarrow \infty$, we get that as $i \rightarrow \infty$,

$$\sup_{t \in [0, T]} \|(\phi^{(i)} - \phi, \partial_t \phi^{(i)} - \partial_t \phi)\|_{H^1(\mathbb{R}^n)}(t) \rightarrow 0.$$

To obtain the result in general for $1 \leq s < k$, simply observe that

$$\begin{aligned} & \sup_{t \in [0, T]} \|(\phi^{(i)} - \phi, \partial_t \phi^{(i)} - \partial_t \phi)\|_{H^s(\mathbb{R}^n)}(t) \\ &= \sup_{t \in [0, T]} \left(\|(\phi^{(i)} - \phi, \partial_t \phi^{(i)} - \partial_t \phi)\|_{H^1(\mathbb{R}^n)}(t) \right)^{\frac{k-s}{k-1}} \left(\|(\phi^{(i)} - \phi, \partial_t \phi^{(i)} - \partial_t \phi)\|_{H^k(\mathbb{R}^n)}(t) \right)^{\frac{s-1}{k-1}} \rightarrow 0. \end{aligned}$$

- (3) In view of part (4) it suffices to assume $k = n + 2$. We proceed by contradiction. Assume that $T_* < \infty$ but

$$\limsup_{t \rightarrow T_*} \sum_{|\alpha| \leq \lfloor \frac{n}{2} + 1 \rfloor} \|\partial_x^\alpha \phi\|_{L^\infty(\mathbb{R}^n)}(t) < \infty.$$

In particular, this implies that there exists a constant $D > 0$ such that

$$\sup_{t \in [0, T_*]} \sum_{|\alpha| \leq \lfloor \frac{n}{2} + 1 \rfloor} \|\partial_x^\alpha \phi\|_{L^\infty(\mathbb{R}^n)}(t) \leq D.$$

Now apply energy estimates to the nonlinear equation for ϕ to get

$$\begin{aligned} & \sup_{t \in [0, T]} \|(\phi, \partial_t \phi)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)} \\ & \leq C \left(\|(\phi_0, \phi_1)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)} + \int_0^T \left(\|F\|_{H^{k-1}(\mathbb{R}^n)}(t) + \sum_{\substack{|\gamma|+|\sigma| \leq k+1 \\ 1 \leq |\gamma|, |\sigma| \leq k}} \sum_{\alpha, \beta} \|\partial^\gamma (a^{\alpha\beta}(\phi)) \partial^\sigma \phi\|_{L^2}(t) \right) dt \right) \\ & \quad \times \exp \left(C \int_0^T \|\partial a(\phi)\|_{L^\infty(\mathbb{R}^n)}(t) dt \right) \end{aligned}$$

for every $T \leq T_*$, where $C = C(T_*)$. Now, we bound F and ∂a as in the proof of part (1) of this theorem. More precisely, we have for some $C = C(D)$ such that

$$\begin{aligned} & \sum_{0 \leq |\alpha| \leq k-1} \|\partial^\alpha F(\phi, \partial\phi)\|_{L^2}(t) \\ & \leq C \left(\sum_{0 \leq |\alpha| \leq k} \|\partial^\alpha \phi\|_{L^2}(t) + \sum_{0 \leq |\alpha_1| + |\alpha_2| \leq k-1} \|\partial \partial^{\alpha_1} \phi \partial \partial^{\alpha_2} \phi\|_{L^2}(t) \right. \\ & \quad \left. + \text{cubic terms} + \dots + \text{terms of order } k-1 \right) \end{aligned}$$

Look for instance at the quadratic term. The key point is that either $|\alpha_1|$ or $|\alpha_2| \leq \frac{k-1}{2}$. Without loss of generality, let's say $|\alpha_1| \leq \frac{k-1}{2}$. Since $k = n+2$, $\frac{k-1}{2} \leq \lfloor \frac{n}{2} + 1 \rfloor$, we have

$$\begin{aligned} & \|\partial \partial^{\alpha_1} \phi \partial \partial^{\alpha_2} \phi\|_{L^2(\mathbb{R}^n)}(t) \\ & \leq C \left(\sum_{1 \leq |\alpha_1| \leq \lfloor \frac{n}{2} + 1 \rfloor} \|\partial^{\alpha_1} \phi\|_{L^\infty(\mathbb{R}^n)}(t) \left(\sum_{1 \leq |\alpha_2| \leq k} \|\partial^{\alpha_2} \phi\|_{L^2(\mathbb{R}^n)}(t) \right) \right) \\ & \leq CD \left(\sum_{1 \leq |\alpha_2| \leq k} \|\partial^{\alpha_2} \phi\|_{L^2(\mathbb{R}^n)}(t) \right). \end{aligned}$$

The higher order terms can be treated in an analogous manner to show that every term is at most linear in

$$\sum_{1 \leq |\alpha_2| \leq k} \|\partial^{\alpha_2} \phi\|_{L^2(\mathbb{R}^n)}(t),$$

multiplied by some polynomials in D . A similar argument can also be used to treat the terms

$$\int_0^T \sum_{\substack{|\gamma| + |\sigma| \leq k+1 \\ 1 \leq |\gamma|, |\sigma| \leq k}} \sum_{\alpha, \beta} \|\partial^\gamma (a^{\alpha\beta}(\phi)) \partial^\sigma \phi\|_{L^2}(t) dt$$

and

$$\int_0^T \|\partial a(\phi)\|_{L^\infty(\mathbb{R}^n)}(t) dt.$$

Therefore, as a consequence, we have

$$\begin{aligned} & \sup_{t \in [0, T]} \|(\phi, \partial_t \phi)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)} \\ & \leq C(D, T_*) \left(\|(\phi_0, \phi_1)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)} + \int_0^T (1+D)^k \|\phi\|_{H^k(\mathbb{R}^n)} dt \right). \end{aligned}$$

By Gronwall's lemma, we get

$$\begin{aligned} & \sup_{t \in [0, T_*]} \|(\phi, \partial_t \phi)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)} \\ & \leq C(D, T_*) \|(\phi_0, \phi_1)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)} \times \exp(CT_*(1+D)^{k-1}). \end{aligned}$$

Consider any sequence of times t_n such that $t_n \rightarrow T_*$. The above estimates show that

$$\|(\phi, \partial_t \phi)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)}(t_n)$$

is uniformly bounded. Now, part (1) of this theorem guarantees that we can find a time of existence T , depending only on $\|(\phi_0, \phi_1)\|_{H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)}$, T_* and D (most importantly independent of $n!$) such that the equation can be solved in $[t_n, t_n + T] \times \mathbb{R}^n$. By taking n sufficiently large, we can therefore extend the solution beyond T_* , contradicting the maximality of T_* !

- (4) We first show the statement for persistence of H^m regularity. We proceed inductively in m . Assume that the conclusion holds for some $m - 1 \geq n + 2$. We will show that conclusion for m . Again, we use energy estimates:

$$\begin{aligned} & \sup_{t \in [0, T]} \|(\phi, \partial_t \phi)\|_{H^m(\mathbb{R}^n) \times H^{m-1}(\mathbb{R}^n)} \\ & \leq C \left(\|(\phi_0, \phi_1)\|_{H^m(\mathbb{R}^n) \times H^{m-1}(\mathbb{R}^n)} + \int_0^T (\|F\|_{H^{m-1}(\mathbb{R}^n)}(t) + \|\partial a\|_{H^{m-1}(\mathbb{R}^n)}(t)) dt \right) \end{aligned}$$

Our goal is to show that in fact

$$\|F\|_{H^{m-1}(\mathbb{R}^n)}(t) + \sum_{\substack{|\gamma|+|\sigma| \leq k+1 \\ 1 \leq |\gamma|, |\sigma| \leq k}} \sum_{\alpha, \beta} \|\partial^\gamma (a^{\alpha\beta}(\phi)) \partial^\sigma \phi\|_{L^2}(t)$$

can be estimated *linearly* in terms of $\sup_{t \in [0, T]} \|(\phi, \partial_t \phi)\|_{H^m(\mathbb{R}^n) \times H^{m-1}(\mathbb{R}^n)}$, which then allows us to apply Gronwall's lemma to obtain the desired estimates in $H^m(\mathbb{R}^n) \times H^{m-1}(\mathbb{R}^n)$. This is possible since m is sufficiently large, and we can use Sobolev embedding together with the induction hypothesis to control the lower order term. More precisely, we have

$$\begin{aligned} & \sum_{0 \leq |\alpha| \leq m-1} \|\partial^\alpha F(\phi, \partial \phi)\|_{L^2}(t) \\ & \leq C \left(\sum_{0 \leq |\alpha| \leq m} \|\partial^\alpha \phi\|_{L^2}(t) + \sum_{0 \leq |\alpha_1| + |\alpha_2| \leq m-1} \|\partial \partial^{\alpha_1} \phi \partial \partial^{\alpha_2} \phi\|_{L^2}(t) \right. \\ & \quad \left. + \text{cubic terms} + \dots + \text{terms of order } k-1 \right) \end{aligned}$$

We look for instance at the quadratic term. We assume without loss of generality that $|\alpha_1| \leq \frac{m+1}{2}$. On the other hand, since $m > n + 2$, we have $\frac{m+1}{2} + \frac{n}{2} < m - 1$. Therefore,

$$\begin{aligned} & \sum_{0 \leq |\alpha_1| + |\alpha_2| \leq m-1} \|\partial \partial^{\alpha_1} \phi \partial \partial^{\alpha_2} \phi\|_{L^2(\mathbb{R}^n)}(t) \\ & \leq C \left(\sum_{0 \leq |\alpha_1| \leq \frac{m+1}{2}} \|\partial^{\alpha_1} \phi\|_{L^\infty(\mathbb{R}^n)}(t) \right) \left(\sum_{0 \leq |\alpha_2| \leq m} \|\partial^{\alpha_2} \phi\|_{L^2(\mathbb{R}^n)}(t) \right) \\ & \leq C \|\phi\|_{H^{m-1}(\mathbb{R}^n)}(t) \|\phi\|_{H^m(\mathbb{R}^n)}(t). \end{aligned}$$

By the induction hypothesis, for every $T < T_*$, we have

$$\sup_{t \in [0, T]} \|\phi\|_{H^{m-1}(\mathbb{R}^n)}(t) \leq C$$

and this term is indeed linear in $\|\phi\|_{H^m(\mathbb{R}^n)}(t)$. The term with a can be treated similarly. Therefore, for every fixed $T < T_*$, the following holds for every $t \in [0, T]$:

$$\begin{aligned} & \sup_{s \in [0, t]} \|(\phi, \partial_t \phi)\|_{H^m(\mathbb{R}^n) \times H^{m-1}(\mathbb{R}^n)} \\ & \leq C(T) \left(\|(\phi_0, \phi_1)\|_{H^m(\mathbb{R}^n) \times H^{m-1}(\mathbb{R}^n)} + \int_0^t \sup_{s \in [0, t]} \|(\phi, \partial_t \phi)\|_{H^m(\mathbb{R}^n) \times H^{m-1}(\mathbb{R}^n)}(s) ds \right). \end{aligned}$$

Gronwall's lemma implies the desired conclusion. □

Remark 6.3. We mention that the conclusion of part (3) of Theorem 6.1 can be improved to

$$\limsup_{t \rightarrow T_*} \sum_{|\alpha| \leq 1} \|\partial_x^\alpha \phi\|_{L^\infty(\mathbb{R}^n)}(t) \rightarrow \infty.$$

We leave this as an exercise (see Example Sheet 2).

7. GLOBAL REGULARITY FOR SUBCRITICAL EQUATIONS

We showed last time that there is a very general local theory for nonlinear wave equations. In general, of course there are many different phenomena associated to the long time behaviour. Nevertheless, there is a class of so-called subcritical equations, such that there is a coercive monotonic quantity above scaling (see discussions below), for which the problem of global regularity for general data is tractable. We look at one particular example here.

Consider the equation

$$(7.1) \quad \begin{cases} \square\phi = |\phi|^2\phi \\ (\phi, \partial_t\phi)|_{\{t=0\}} = (\phi_0, \phi_1). \end{cases}$$

We first show that

Proposition 7.1. *As long as the solution remains sufficiently regular, the following quantity is independent of time:*

$$E(t) := \int_{\mathbb{R}^3} \frac{1}{2}((\partial_t\phi)^2 + \sum_{i=1}^3(\partial_i\phi)^2)(t, x) + \frac{1}{4}|\phi|^4(t, x)dx.$$

Proof. This is a direct computation. \square

Notice that the conservation of a positive quantity alone does not necessarily imply that the solution is globally regular. It is important to show that this quantity in fact gives sufficiently strong control over the solution. In the context of this equation, this is in fact manifested in the following Sobolev inequality:

Proposition 7.2. *There exists a constant $C > 0$ such that for every $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$, we have*

$$\|\phi\|_{L^6(\mathbb{R}^3)} \leq C\|\phi\|_{\dot{H}^1(\mathbb{R}^3)}.$$

We are now ready to show that

Theorem 7.3. *Assume $(\phi_0, \phi_1) \in (H^5 \times H^4) \cap (C^\infty \times C^\infty)$. Then (7.1) has a global-in-time smooth solution.*

Proof. We apply the local existence theorem to conclude that there is a smooth local-in-time solution. According to Theorem 6.1, it suffices to show that for every T , we have the bound

$$\sum_{|\alpha| \leq 2} \|\partial^\alpha \phi\|_{L^\infty([0, T] \times \mathbb{R}^3)} \leq C(T).$$

By Sobolev embedding (Theorem 5.4), it in turn suffices to prove

$$\|\phi\|_{L^\infty([0, T]; H^4(\mathbb{R}^3))} \leq C(T).$$

We do this in two steps. First, we control the H^2 norm.

$$\begin{aligned} & \|(\phi, \partial_t\phi)\|_{L^\infty([0, T]; H^2(\mathbb{R}^3)) \times L^\infty([0, T]; H^1(\mathbb{R}^3))} \\ & \leq C(T) \left(\|(\phi, \partial_t\phi)\|_{H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)} + \int_0^T \|\phi \cdot \phi \cdot \partial\phi\|_{L^2(\mathbb{R}^n)}(t) dt \right) \\ & \leq C(T) \left(\|(\phi, \partial_t\phi)\|_{H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)} + \int_0^T \|\phi\|_{L^6(\mathbb{R}^n)}(t) \|\phi\|_{L^6(\mathbb{R}^n)}(t) \|\partial\phi\|_{L^6(\mathbb{R}^n)}(t) dt \right) \\ & \leq C(T) \left(\|(\phi, \partial_t\phi)\|_{H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)} + \int_0^T (E(t))^2 \|(\phi, \partial_t\phi)\|_{H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)}(t) dt \right). \end{aligned}$$

Since E is a priori controlled using the conservation law, we can apply Gronwall's lemma to obtain

$$\|(\phi, \partial_t\phi)\|_{L^\infty([0, T]; H^2(\mathbb{R}^3)) \times L^\infty([0, T]; H^1(\mathbb{R}^3))} \leq C(T).$$

Here, of course we have abused notation such that this constant $C(T)$ is (exponentially) larger than the constants in the lines above. In particular, this implies that ϕ is bounded in $L^\infty([0, T] \times \mathbb{R}^3)$.

We now proceed to controlling the H^4 norm.

$$\begin{aligned}
& \|(\phi, \partial_t \phi)\|_{L^\infty([0, T]; H^4(\mathbb{R}^3)) \times L^\infty([0, T]; H^3(\mathbb{R}^3))} \\
& \leq C(T) \left(\|(\phi, \partial_t \phi)\|_{H^4(\mathbb{R}^3) \times H^3(\mathbb{R}^3)} + \sum_{|\alpha|+|\beta|+|\gamma|=3} \int_0^T \|\partial^\alpha \phi \cdot \partial^\beta \phi \cdot \partial^\gamma \phi\|_{L^2(\mathbb{R}^n)}(t) dt \right) \\
& \leq C(T) \left(\|(\phi, \partial_t \phi)\|_{H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)} + \int_0^T \left(\sum_{|\alpha|=1} \|\partial^\alpha \phi\|_{L^6(\mathbb{R}^n)}(t) \right)^3 dt \right. \\
& \quad + \int_0^T (\|\phi\|_{L^\infty(\mathbb{R}^n)}(t))^2 \sum_{|\alpha|=3} \|\partial^\alpha \phi\|_{L^2(\mathbb{R}^n)}(t) dt \\
& \quad \left. + \int_0^T \|\phi\|_{L^6(\mathbb{R}^n)}(t) \left(\sum_{|\alpha|=1} \|\partial^\alpha \phi\|_{L^6(\mathbb{R}^n)}(t) \right) \left(\sum_{|\beta|=2} \|\partial^\beta \phi\|_{L^6(\mathbb{R}^n)}(t) \right) dt \right).
\end{aligned}$$

Now, the key points are that

- (1) ϕ is in L^6 by the conservation law
- (2) ϕ in L^∞ and $\partial\phi$ in L^6 using the H^2 control that we have obtained above.

Therefore, the above estimate is *linear* in

$$\|(\phi, \partial_t \phi)\|_{H^4(\mathbb{R}^3) \times H^3(\mathbb{R}^3)}(t),$$

i.e.,

$$\begin{aligned}
& \|(\phi, \partial_t \phi)\|_{L^\infty([0, T]; H^4(\mathbb{R}^3)) \times L^\infty([0, T]; H^3(\mathbb{R}^3))} \\
& \leq C(E, T) \left(\|(\phi, \partial_t \phi)\|_{H^4(\mathbb{R}^3) \times H^3(\mathbb{R}^3)} + \int_0^T \|(\phi, \partial_t \phi)\|_{H^4(\mathbb{R}^3) \times H^3(\mathbb{R}^3)}(t) dt \right).
\end{aligned}$$

We can therefore conclude using Gronwall's lemma. \square

Remark 7.4. Another perhaps more direct way to prove this theorem is to reprove the local existence theorem and show that if we perform Picard's iteration in $L^\infty([0, T]; H^1) \times L^\infty([0, T]; L^2)$, we can show that there exists a solution in $[0, T] \times \mathbb{R}^3$, where T **depends only on** $\|\phi_0\|_{\dot{H}^1(\mathbb{R}^3)}$ **and** $\|\phi_1\|_{L^2(\mathbb{R}^3)}$. Moreover, we can also show as in the local existence theorem that as long as an H^1 solution exists, then higher regularity is propagated. This allows us to conclude using the conservation law.

We now briefly discuss the concepts of scaling and criticality. Notice that the above PDE has two features. Firstly, it has a conservation law that controls the \dot{H}^1 norm. Secondly, it is invariant with respect to the scaling

$$\phi_\lambda(t, x) = \lambda \phi(\lambda t, \lambda x),$$

i.e., if $\phi(t, x)$ is a solution, then $\phi_\lambda(t, x)$ is also a solution. Now, here is the key point: if we scale to smaller length scales, i.e., we take $\lambda \rightarrow \infty$, then the \dot{H}^1 norm $\rightarrow \infty$. This shows that the conservation law (or more generally a monotonic quantity) is all the more useful at smaller length scales. We call this property **subcritical**. On the other hand, we say that an equation is **supercritical** if the opposite is true and is **critical** if the conservation law scales in exactly the same way as the scaling of the equation.

The concept of criticality is extremely important to predict whether an equation has global regular large data solutions. Most of the time, subcritical equations have a global regular solution. Critical equations, lying at the threshold of subcritical and supercritical equations, exhibit a large array of phenomenon - some of them always have globally regular solutions while some possess solutions that exhibit finite time blow up. On the other hand, very little is known at all about supercritical equations!

Returning to the equation that we were considering, if we look at a more general class of equations

$$\square\phi = |\phi|^{p-1}\phi$$

in $\mathbb{R} \times \mathbb{R}^3$, we see that the \dot{H}^1 norm is always controlled by the conservation law while the equation is invariant under the scaling

$$\phi_\lambda(t, x) = \lambda^{\frac{2}{p-1}} \phi(\lambda t, \lambda x).$$

The equation is therefore (**Exercise**) subcritical if $p < 5$; critical if $p = 5$ and supercritical if $p > 5$. Indeed in the first two cases, it is known that regular initial data give rise to globally regular solutions. (We see moreover that the full range of $1 \leq p \leq 3$ can be treated as above⁵. However, the case $3 < p \leq 5$ requires techniques that we will not have time to cover in the lectures. See the Example Sheet for further discussions.) On the other hand, almost nothing is known regarding large data solutions in the supercritical case!

We end this section with a discussion on the wave map equation from $\mathbb{R} \times \mathbb{R}^n$ to \mathbb{S}^m . Recall that it is given by

$$\square \phi = -\phi(\partial_t \phi^t \partial_t \phi - \sum_{i=1}^n \partial_i \phi^t \partial_i \phi),$$

with initial data $|\phi_0|^2 = 1$ and $\phi_1^t \phi_0 = 0$. For all $n \geq 1$ and $m \geq 1$, we have the following conservation law:

Proposition 7.5. *As long as the solution remains sufficiently regular, the following quantity*

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^n} (|\partial_t \phi|^2 + |\nabla \phi|^2)(t, x) dx$$

is independent of t .

Proof. Using the fact $|\phi|^2 = 1$, we have $\phi^t \partial_\alpha \phi$ for $\alpha = 0, 1, \dots, n$. Therefore, using the wave map equations, we have

$$(7.2) \quad -\frac{1}{2} \partial_t (\partial_t \phi^t \partial_t \phi + \sum_{i=1}^n \partial_i \phi^t \partial_i \phi) + \sum_{i=1}^n \partial_i (\partial_t \phi \partial_i \phi) = 0.$$

Integrating by parts in the region between any two times, we obtain the desired conclusion. \square

On the other hand, the wave map equation is invariant under the scaling

$$\phi_\lambda(t, x) = \phi(\lambda t, \lambda x).$$

Notice that ϕ_λ is indeed a map to the sphere for all λ . In particular, the $n = 1$ case is subcritical; the $n = 2$ case is critical and the $n \geq 3$ case is supercritical. In dimensions $n \geq 2$, there exist solutions which arise from smooth initial data but blow up in finite time⁶. On the other hand, global regularity holds in the $n = 1$ case.

Proposition 7.6. *$(H^2 \times H^1) \cap (C^\infty \times C^\infty)$ initial data for the 1 + 1-dimensional wave map equation $\mathbb{R}^{1+1} \rightarrow \mathbb{S}^2$ give rise to global-in-time smooth solution.*

Proof. We first make a remark regarding the applicability of Theorem 6.1. Notice that unlike in the setting in Theorem 6.1, the wave map equations are in fact a *system* of wave equations as opposed to a single scalar wave equation. Nevertheless, it is easy to see that the proof of Theorem 6.1 using energy estimates can also be applied in this case after trivial modifications.

Therefore, using this local existence theorem (Theorem 6.1), it suffices to show that

$$\sum_{|\alpha| \leq 1} \|\partial^\alpha \phi\|_{L^\infty(\mathbb{R})} \leq C.$$

When $|\alpha| = 0$, this is already implied by the fact that this is a map to the sphere, i.e., $|\phi|^2 = 1$. It turns out that in 1 + 1 dimensions, the wave map equation admits many more conservation laws - indeed we have infinitely many of them! Notice that (**Exercise**)

$$\square(\partial_t \phi^t \partial_t \phi + \partial_x \phi^t \partial_x \phi) = 0.$$

⁵Note that if $p \in (1, 3)$, the nonlinearity is not smooth. So in principle we cannot apply the local theory that we had. On the other hand, it is not difficult to see that one can construct $L^\infty([0, T]; H^1(\mathbb{R}^3)) \times L^\infty([0, T]; L^2(\mathbb{R}^3))$ directly for these nonlinearities.

⁶Notice while in both the critical and supercritical cases, the existence of blow-up solutions has been exhibited, in general the dynamics is much better understood in the $n = 2$ case.

In particular, this shows that

$$\sum_{|\alpha| \leq 1} \|\partial^\alpha \phi\|_{L^\infty(\mathbb{R})} \leq C.$$

□

Remark 7.7. Using

$$\square(\partial_t \phi^t \partial_t \phi + \partial_x \phi^t \partial_x \phi) = 0,$$

we conclude that for $v = t + x$ and $u = t - x$, $\partial_v(\partial_t \phi^t \partial_t \phi + \partial_x \phi^t \partial_x \phi)$ is independent of u and $\partial_u(\partial_t \phi^t \partial_t \phi + \partial_x \phi^t \partial_x \phi)$ is independent of v ! Hence we have uncountably many conservation laws! Equations with infinitely many conserved quantities can typically called **completely integrable**. Many techniques have been developed to give very precise information about the solutions to completely integrable equations. However, of course these techniques are very special and can only be applied to the very restricted class of completely integrable equations.

We now give an alternative proof of the global regularity of the $(1+1)$ -dimensional wave map which uses the **characteristic energy**, but not the additional conservation laws.

Proof. We first prove the boundedness of the characteristic energy. We start with (7.2) and integrate by parts in the region

$$\{0 \leq t \leq T\} \cap \{v \leq v_0\}.$$

This shows that as long as the solution remains regular in $t \leq T$, the following quantity is a priori bounded:

$$\int_{-v_0}^{2T-v_0} |\partial_u \phi|^2(u, v_0) du \leq C \|(\phi_0, \phi_1)\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})}.$$

Similarly, integrating by parts in the region

$$\{0 \leq t \leq T\} \cap \{u \leq u_0\},$$

we get

$$\int_{-u_0}^{2T-u_0} |\partial_v \phi|^2(u_0, v) dv \leq C \|(\phi_0, \phi_1)\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})}.$$

Since u_0, v_0 are arbitrary, we have

$$(7.3) \quad \sup_v \int_{-v}^{2T-v} |\partial_u \phi|^2(u, v) du + \sup_{u_0} \int_{-u_0}^{2T-u_0} |\partial_v \phi|^2(u, v) dv \leq C \|(\phi_0, \phi_1)\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})}.$$

We now use this and the equation to control the derivatives of ϕ in L^∞ . More precisely, we write the equation as

$$-4\partial_u \partial_v \phi = -\phi(\partial_t \phi^t \partial_t \phi - \sum_{i=1}^n \partial_i \phi^t \partial_i \phi) = -4\phi \partial_u \phi^t \partial_v \phi.$$

We can integrate in the u direction to control $\partial_v \phi$, namely, for $u + v \leq 2T$, we have

$$\sup_{v \in \mathbb{R}} |\partial_v \phi|(u, v) \leq C \|(\phi_0, \phi_1)\|_{W^{1,\infty}(\mathbb{R}) \times L^\infty(\mathbb{R})} + \sup_v \int_{-v}^u |\partial_u \phi| |\partial_v \phi|(u', v) du'.$$

Gronwall's inequality implies that

$$\begin{aligned} \sup_{u, v \in \mathbb{R}} |\partial_v \phi|(u, v) &\leq C \|(\phi_0, \phi_1)\|_{W^{1,\infty}(\mathbb{R}) \times L^\infty(\mathbb{R})} \exp\left(C \sup_v \int_{-v}^u |\partial_u \phi|(u', v) du'\right) \\ &\leq C \|(\phi_0, \phi_1)\|_{W^{1,\infty}(\mathbb{R}) \times L^\infty(\mathbb{R})} \exp\left(CT^{\frac{1}{2}} \|(\phi_0, \phi_1)\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})}\right), \end{aligned}$$

where in the last step we have used (7.3). In a similar manner, we can also use (7.3) to obtain

$$\sup_{u, v \in \mathbb{R}} |\partial_u \phi|(u, v) \leq C \|(\phi_0, \phi_1)\|_{W^{1,\infty}(\mathbb{R}) \times L^\infty(\mathbb{R})} \exp\left(CT^{\frac{1}{2}} \|(\phi_0, \phi_1)\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})}\right).$$

Since we have controlled the first derivatives of ϕ on any time interval $[0, T]$, we apply Theorem 6.1 to conclude the proof. □

8. EINSTEIN VACUUM EQUATIONS

We now return to the local theory for nonlinear wave equations and show that it gives a local theory for the Einstein vacuum equations

$$Ric(g) = 0.$$

As we mentioned in the introduction, the Einstein vacuum equations a priori do not look like a system of wave equation. We first recall some basic notions in differential geometry and derive the formula. (Note that repeated indices are always summed over!)

Definition 8.1. Given a metric g , define the Levi-Civita connection ∇ by

$$\nabla_{X^\alpha} \frac{\partial}{\partial x^\alpha} Y^\beta = \frac{\partial Y^\beta}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} + \Gamma_{\alpha\beta}^\mu X^\alpha Y^\beta \frac{\partial}{\partial x^\mu},$$

where $\Gamma_{\alpha\beta}^\mu$ is the Christoffel symbols given by

$$\Gamma_{\alpha\beta}^\mu = \frac{1}{2}(g^{-1})^{\mu\nu} \left(\frac{\partial}{\partial x^\beta} g_{\alpha\nu} + \frac{\partial}{\partial x^\alpha} g_{\beta\nu} - \frac{\partial}{\partial x^\nu} g_{\alpha\beta} \right).$$

Definition 8.2. Define the Riemann curvature tensor R by

$$R^\gamma{}_{\alpha\mu\nu} \frac{\partial}{\partial x^\gamma} = \nabla_{\frac{\partial}{\partial x^\mu}} \nabla_{\frac{\partial}{\partial x^\nu}} \frac{\partial}{\partial x^\alpha} - \nabla_{\frac{\partial}{\partial x^\nu}} \nabla_{\frac{\partial}{\partial x^\mu}} \frac{\partial}{\partial x^\alpha}$$

and

$$R_{\beta\alpha\mu\nu} = g_{\beta\gamma} R^\gamma{}_{\alpha\mu\nu}.$$

We recall (without proof) some standard properties of the Riemann curvature tensor:

Proposition 8.3. *The Riemann curvature tensor satisfies the following properties:*

(1)

$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} = -R_{\mu\nu\beta\alpha};$$

(2)

$$R_{\mu\nu\alpha\beta} + R_{\alpha\mu\nu\beta} + R_{\nu\alpha\mu\beta} = 0;$$

(3)

$$\nabla_\sigma R_{\mu\nu\alpha\beta} + \nabla_\mu R_{\nu\sigma\alpha\beta} + \nabla_\nu R_{\sigma\mu\alpha\beta} = 0.$$

Definition 8.4. Define the Ricci curvature tensor by

$$Ric_{\mu\nu} = (g^{-1})^{\alpha\beta} R_{\mu\alpha\nu\beta}.$$

We compute

$$\nabla_{\frac{\partial}{\partial x^\mu}} \nabla_{\frac{\partial}{\partial x^\nu}} \frac{\partial}{\partial x^\alpha} = \nabla_{\frac{\partial}{\partial x^\mu}} \left(\Gamma_{\nu\alpha}^\rho \frac{\partial}{\partial x^\rho} \right) = \left(\frac{\partial}{\partial x^\mu} \Gamma_{\nu\alpha}^\rho \right) + \Gamma_{\nu\alpha}^\sigma \Gamma_{\mu\sigma}^\rho \frac{\partial}{\partial x^\rho}.$$

We will only consider terms that has two derivatives of the metric.

$$R_{\mu\nu\alpha}{}^\gamma = -\frac{\partial}{\partial x^\mu} \Gamma_{\nu\alpha}^\gamma + \frac{\partial}{\partial x^\nu} \Gamma_{\mu\alpha}^\gamma + \dots$$

and thus the Ricci curvature tensor is given by

$$Ric_{\mu\nu} = -\frac{\partial}{\partial x^\mu} \Gamma_{\nu}^\gamma + \frac{\partial}{\partial x^\nu} \Gamma_{\mu}^\gamma + \dots$$

We can show that in a local coordinate system, the Einstein vacuum equations read

(8.1)

$$0 = Ric(g)_{\mu\nu} = -\frac{1}{2}(g^{-1})^{\alpha\beta} \partial_{\alpha\beta}^2 g_{\mu\nu} - \frac{1}{2}(g^{-1})^{\alpha\beta} \partial_{\mu\nu}^2 g_{\alpha\beta} + \frac{1}{2}(g^{-1})^{\alpha\beta} \partial_{\alpha\nu}^2 g_{\beta\mu} + \frac{1}{2}(g^{-1})^{\alpha\beta} \partial_{\beta\mu}^2 g_{\alpha\nu} + F_{\mu\nu}(g, \partial g),$$

where $F_{\mu\nu}(g, \partial g)$ is a function of g and its derivatives. In particular, if the second, third and fourth terms are absent, then this is a wave equation and the general local theory for quasilinear equations can be applied. On the other hand, for a particular choice of coordinate system which satisfies the so-called **wave coordinate condition**

(8.2)

$$\square_g x^\alpha = \frac{1}{\sqrt{-g}} \partial_\mu ((g^{-1})^{\mu\nu} \sqrt{-g} \partial_\nu x^\alpha) = 0$$

then indeed the second, third and fourth terms can be re-written as terms having at most one derivatives of g .

To see this, we use the matrix identities

$$\partial_\alpha (g^{-1})^{\mu\nu} = -(g^{-1})^{\mu\beta} (g^{-1})^{\sigma\nu} \partial_\alpha g_{\sigma\beta}$$

and

$$\partial_\alpha \log \det g = (g^{-1})^{\mu\nu} \partial_\alpha g_{\mu\nu}.$$

The wave coordinate condition (8.2) implies

$$0 = \partial_\mu ((g^{-1})^{\mu\nu} \sqrt{-g}) = -\sqrt{-g} \left((g^{-1})^{\mu\alpha} (g^{-1})^{\nu\beta} \partial_\mu g_{\alpha\beta} - \frac{1}{2} (g^{-1})^{\mu\nu} (g^{-1})^{\alpha\beta} \partial_\mu g_{\alpha\beta} \right),$$

which, after contracting with $g_{\nu\sigma}$, in turn implies

$$(g^{-1})^{\mu\alpha} \partial_\mu g_{\alpha\sigma} = \frac{1}{2} (g^{-1})^{\alpha\beta} \partial_\sigma g_{\alpha\beta}.$$

Now define λ_σ to be

$$\lambda_\sigma = (g^{-1})^{\mu\alpha} \partial_\mu g_{\alpha\sigma} - \frac{1}{2} (g^{-1})^{\alpha\beta} \partial_\sigma g_{\alpha\beta}$$

so that λ_σ vanishes if (8.2) holds. Using this, we derive

$$\begin{aligned} & -\frac{1}{2} (g^{-1})^{\alpha\beta} \partial_{\mu\nu}^2 g_{\alpha\beta} \\ &= -\frac{1}{2} \partial_\mu ((g^{-1})^{\alpha\beta} \partial_\nu g_{\alpha\beta}) + \frac{1}{2} (\partial_\mu (g^{-1})^{\alpha\beta}) \partial_\nu g_{\alpha\beta} \\ &= -\partial_\mu ((g^{-1})^{\alpha\beta} \partial_\beta g_{\alpha\nu}) + \partial_\mu \lambda_\sigma + \frac{1}{2} (\partial_\mu (g^{-1})^{\alpha\beta}) \partial_\nu g_{\alpha\beta} \\ &= -(g^{-1})^{\alpha\beta} \partial_{\beta\mu}^2 g_{\alpha\nu} - (\partial_\mu (g^{-1})^{\alpha\beta}) \partial_\beta g_{\alpha\nu} + \frac{1}{2} (\partial_\mu (g^{-1})^{\alpha\beta}) \partial_\nu g_{\alpha\beta} + \partial_\mu \lambda_\sigma \\ &= -(g^{-1})^{\alpha\beta} \partial_{\beta\mu}^2 g_{\alpha\nu} + (g^{-1})^{\alpha\sigma} (g^{-1})^{\beta\rho} \partial_\mu g_{\sigma\rho} \partial_\beta g_{\alpha\nu} - \frac{1}{2} (g^{-1})^{\alpha\sigma} (g^{-1})^{\beta\rho} \partial_\mu g_{\sigma\rho} \partial_\nu g_{\alpha\beta} + \partial_\mu \lambda_\sigma. \end{aligned}$$

In a completely analogous manner, we also have

$$\begin{aligned} & -\frac{1}{2} (g^{-1})^{\alpha\beta} \partial_{\mu\nu}^2 g_{\alpha\beta} \\ &= -(g^{-1})^{\alpha\beta} \partial_{\alpha\nu}^2 g_{\beta\mu} + (g^{-1})^{\alpha\sigma} (g^{-1})^{\beta\rho} \partial_\nu g_{\sigma\rho} \partial_\alpha g_{\beta\mu} - \frac{1}{2} (g^{-1})^{\alpha\sigma} (g^{-1})^{\beta\rho} \partial_\mu g_{\sigma\rho} \partial_\nu g_{\alpha\beta} + \partial_\sigma \lambda_\mu. \end{aligned}$$

Using the previous calculations, we have thus shown the following result:

Proposition 8.5. *Define the reduced Ricci curvature to be*

$$\begin{aligned} \widetilde{Ric}(g)_{\mu\nu} &:= -\frac{1}{2} (g^{-1})^{\alpha\beta} \partial_{\alpha\beta}^2 g_{\mu\nu} + \frac{1}{2} (g^{-1})^{\alpha\sigma} (g^{-1})^{\beta\rho} \partial_\mu g_{\sigma\rho} \partial_\beta g_{\alpha\nu} + \frac{1}{2} (g^{-1})^{\alpha\sigma} (g^{-1})^{\beta\rho} \partial_\nu g_{\sigma\rho} \partial_\alpha g_{\beta\mu} \\ &\quad - \frac{1}{2} (g^{-1})^{\alpha\sigma} (g^{-1})^{\beta\rho} \partial_\mu g_{\sigma\rho} \partial_\nu g_{\alpha\beta} + F_{\mu\nu}(g, \partial g), \end{aligned}$$

where F is as in (8.1). Then

$$\widetilde{Ric}(g)_{\mu\nu} = Ric(g)_{\mu\nu} + \frac{1}{2} \partial_\mu \lambda_\nu + \frac{1}{2} \partial_\nu \lambda_\mu.$$

In other words, if the wave coordinate condition holds, the Einstein vacuum equations become the reduced Einstein vacuum equations, which is a system of nonlinear wave equations. However, we of course need to guarantee that the wave coordinate condition holds! The strategy, introduced by Choquet-Bruhat, is to construct local solutions to the reduced Einstein vacuum equations and show that in fact if the wave coordinate condition (8.2) holds initially, then it is propagated by the flow. One of the key observations is that λ_σ in fact satisfies a wave equation.

Proposition 8.6. *Given a Lorentzian metric g such that the reduced Einstein vacuum equations are satisfied, i.e., $\widetilde{Ric}(g) = 0$. Then λ satisfies a wave equation:*

$$\frac{1}{2} (g^{-1})^{\sigma\mu} \partial_{\sigma\mu}^2 \lambda_\nu + c_\nu^{\alpha\beta} \partial_\alpha \lambda_\beta = 0,$$

where $c_\nu^{\alpha\beta}$ are smooth functions of g and its derivatives.

Proof. The proof is based on the following identity:

$$(8.3) \quad \nabla^\mu (Ric_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 0.$$

To see that this identity holds, first we use the second Bianchi identity (part 3 in Proposition 8.3) to get that

$$\begin{aligned} \nabla^\mu R_{\mu\nu\alpha\beta} &= (g^{-1})^{\mu\sigma} \nabla_\sigma R_{\mu\nu\alpha\beta} = - (g^{-1})^{\mu\sigma} \nabla_\alpha R_{\mu\nu\beta\sigma} - (g^{-1})^{\mu\sigma} \nabla_\beta R_{\mu\nu\sigma\alpha} \\ &= \nabla_\alpha Ric_{\beta\nu} - \nabla_\beta Ric_{\alpha\nu}. \end{aligned}$$

Taking the trace of this identity in α, ν , we get (8.3).

We now use this to derive an equation for λ . First, the vanishing of $\widetilde{Ric}(g)$ implies that

$$0 = Ric(g)_{\mu\nu} + \frac{1}{2}\partial_\mu\lambda_\nu + \frac{1}{2}\partial_\nu\lambda_\mu$$

and in particular after taking the trace, we have

$$0 = R + (g^{-1})^{\mu\nu}\partial_\mu\lambda_\nu.$$

Therefore, by (8.3), we have

$$\begin{aligned} 0 &= \nabla^\mu (Ric(g)_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) \\ &= - (g^{-1})^{\sigma\mu} \partial_\sigma (-\frac{1}{2}\partial_\mu\lambda_\nu - \frac{1}{2}\partial_\nu\lambda_\mu + \frac{1}{2}g_{\mu\nu}(g^{-1})^{\alpha\beta}\partial_\alpha\lambda_\beta) \\ &\quad - (g^{-1})^{\sigma\mu} \Gamma_{\sigma\mu}^\delta (-\frac{1}{2}\partial_\delta\lambda_\nu - \frac{1}{2}\partial_\nu\lambda_\delta + \frac{1}{2}g_{\delta\nu}(g^{-1})^{\alpha\beta}\partial_\alpha\lambda_\beta) \\ &\quad - (g^{-1})^{\sigma\mu} \Gamma_{\sigma\nu}^\delta (-\frac{1}{2}\partial_\mu\lambda_\delta - \frac{1}{2}\partial_\delta\lambda_\mu + \frac{1}{2}g_{\mu\delta}(g^{-1})^{\alpha\beta}\partial_\alpha\lambda_\beta) \\ &= \frac{1}{2}(g^{-1})^{\sigma\mu} \partial_{\sigma\mu}^2 \lambda_\nu + c_\nu^{\alpha\beta} \partial_\alpha \lambda_\beta, \end{aligned}$$

for some $c_\nu^{\alpha\beta}$ which are smooth functions of g and its derivatives. □

Before we proceed to the main theorem, we first discuss the initial data that have to be posed. Given that we are to solve the reduced Einstein vacuum equations, which are nonlinear wave equations for the metric g , we need to prescribe the initial g and $\partial_t g$. It turns out that we only need to prescribe the metric intrinsic to the initial hypersurface and the second fundamental form. The second fundamental form is defined as

$$\hat{k}_{ij} := \frac{1}{2}(\mathcal{L}_n g)_{ij} := \frac{1}{2}n^\ell \partial_\ell g_{ij} + \frac{1}{2}g_{j\ell} \partial_i n^\ell + \frac{1}{2}g_{i\ell} \partial_j n^\ell,$$

where n is the normal to the initial hypersurface such that $g(n, n) = -1$. The remaining components for g and $\partial_t g$ can be prescribed as coordinate conditions. More precisely, we choose ∂_t to be of unit length -1 and orthogonal to the initial hypersurface, i.e. $g_{00} \upharpoonright_{\{t=0\}} = -1$, $g_{0i} \upharpoonright_{\{t=0\}} = 0$. Under this choice, the second fundamental form k_{ij} becomes

$$\hat{k}_{ij} = \frac{1}{2}\partial_t g_{ij}.$$

Now, the remaining choices of the initial conditions (i.e., $\partial_t g_{00}$ and $\partial_t g_{0i}$) are fixed by the wave coordinate condition. More precisely,

$$0 = \lambda_i = (g^{-1})^{\mu\alpha} \partial_\mu g_{\alpha i} - \frac{1}{2}(g^{-1})^{\alpha\beta} \partial_i g_{\alpha\beta}$$

fixes $\partial_t g_{0i}$ and

$$0 = \lambda_0 = (g^{-1})^{\mu\alpha} \partial_\mu g_{\alpha 0} - \frac{1}{2}(g^{-1})^{\alpha\beta} \partial_0 g_{\alpha\beta}$$

fixes $\partial_t g_{00}$.

Before we proceed, we make a final observation. We compute $(Ric - \frac{1}{2}gR)_{00}$ and Ric_{0i} on the initial hypersurface and note that there are no terms of the form $\partial_t^2 g$! To see that, notice that

$$Ric_{00} = \frac{1}{2}(g^{-1})^{00}\partial_t^2 g_{00} - \frac{1}{2}(g^{-1})^{\alpha\beta}\partial_t^2 g_{\alpha\beta} + \dots$$

On the other hand,

$$(g^{-1})^{\alpha\beta} Ric_{\alpha\beta} = -(g^{-1})^{00}(g^{-1})^{\alpha\beta}\partial_t^2 g_{\alpha\beta} + (g^{-1})^{00}(g^{-1})^{00}\partial_t^2 g_{00} + \dots$$

Hence $(Ric - \frac{1}{2}gR)_{00}$ does not depend on the second time derivatives of g . Similarly for Ric_{0i} .

In other words, if $(Ric - \frac{1}{2}gR)_{00}$ and Ric_{0i} are to vanish, they must be conditions that have to be *imposed* initially. They are called the constraint equations and can be given geometrically as equations on the initial hypersurface:

$$\begin{aligned} \hat{\nabla}_i \hat{k}^i_j - \hat{\nabla}_j \hat{k}^i_i &= 0, \\ \hat{R}(\hat{g}) + (\hat{k}^i_i)^2 - \hat{k}^i_j \hat{k}^j_i &= 0. \end{aligned}$$

Combining the above results, we have thus proved the local existence theorem of Choquet-Bruhat:

Theorem 8.7. *Given initial data $(\hat{g}_{ij}, \hat{k}_{ij})$ on \mathbb{R}^n satisfying the constraint equations and such that $\sum_{i,j} |\hat{g}_{ij} - \delta_{ij}| \leq \frac{1}{4}$, there exists a metric g in $I \times \mathbb{R}^n$ which solves the Einstein vacuum equations such that the induced metric and the induced second fundamental form on $\{0\} \times \mathbb{R}^n$ coincide with \hat{g}_{ij} and \hat{k}_{ij} respectively.*

Proof. Given initial data above, we solve the reduced Einstein vacuum equations, which are a system of nonlinear wave equations. By the local existence theory, there exists a local solution to the reduced Einstein vacuum equations.

To show that this solution is indeed a solution to the Einstein vacuum equations, we need to show that $\lambda_\sigma = 0$. In view of the fact that λ_σ satisfies a wave equation, it suffices to show that $(\lambda_\sigma, \partial_t \lambda_\sigma) \upharpoonright_{\{t=0\}} = (0, 0)$ for $\sigma = 0, 1, 2, 3$.

Of course, we have already set $\lambda_\sigma(t=0) = 0$. To show that $\partial_t \lambda_\sigma(t=0) = 0$, we apply Proposition 8.5. The vanishing of $\widetilde{Ric}(g)_{\mu\nu}$ gives

$$0 = Ric(g)_{\mu\nu} + \frac{1}{2}\partial_\mu \lambda_\nu + \frac{1}{2}\partial_\nu \lambda_\mu,$$

which also implies

$$R = -(g^{-1})^{\mu\nu} \partial_\mu \lambda_\nu.$$

Since the initial data (\hat{g}, \hat{k}) satisfy the constraints $Ric(g)_{i0} \upharpoonright_{\{t=0\}} = 0$ and $(Ric(g)_{00} - \frac{1}{2}Rg_{00}) \upharpoonright_{\{t=0\}} = 0$, we have

$$\frac{1}{2}\partial_t \lambda_i \upharpoonright_{\{t=0\}} = -\frac{1}{2}\partial_i \lambda_0 \upharpoonright_{\{t=0\}} = 0$$

and

$$(\partial_t \lambda_0 + \frac{1}{2}(g^{-1})^{\mu\nu} \partial_\mu \lambda_\nu) \upharpoonright_{\{t=0\}} = 0.$$

The latter condition can be rewritten to give

$$\frac{1}{2}\partial_t \lambda_0 \upharpoonright_{\{t=0\}} = -\frac{1}{2}(g^{-1})^{ij} \partial_i \lambda_j \upharpoonright_{\{t=0\}} = 0.$$

This concludes the proof. \square

Remark 8.8. In the proof, we needed the assumption that $\sum_{i,j} |\hat{g}_{ij} - \delta_{ij}| \leq \frac{1}{4}$. This can be removed by localizing in neighbourhoods where the metric \hat{g}_{ij} is close to some constant coefficient metric. We then change coordinates so that this is close to δ_{ij} and apply the theorem above. We then use finite speed of propagation for the equation.

Remark 8.9. In the proof of Theorem 6.1, we required that the coefficients satisfy $\sum_{\mu,\nu} |(g^{-1})^{\mu\nu} - m^{\mu\nu}| \leq \frac{1}{2}$ everywhere while our assumptions in Theorem 8.7 only guarantee that this holds on the initial slice. Nevertheless, one can show that by choosing T sufficiently small in the proof of Theorem 6.1, we can in fact guarantee that $\sum_{\mu,\nu} |(g^{-1})^{\mu\nu} - m^{\mu\nu}| \leq \frac{1}{2}$ everywhere in $I \times \mathbb{R}^n$.

Remark 8.10. Of course, more generally, the initial data are not required to be posed on a n -dimensional manifold that has the topology of \mathbb{R}^n . Indeed, initial data can be posed on any Riemannian manifold.

Remark 8.11. One may worry whether there are any non-trivial solutions to the constraint equations. Solving the constraint equations turn out to be a subject in its own. It suffices to say for the purpose of this course that there are many solutions to the constraint equations.

Remark 8.12. Finally, let us mention that geometric uniqueness also holds in the sense that given two solutions (\mathcal{M}_1, g_1) and (\mathcal{M}_2, g_2) , there exists open subsets $U_i \subset \mathcal{M}_i$ containing Σ such that (U_1, g_1) and (U_2, g_2) are isometric.

9. DECAY OF SOLUTIONS USING ENERGY METHODS

After a discussion of local theory for general nonlinear wave equations, we return to the linear wave equation on Minkowski spacetime:

$$\square\phi = 0.$$

Recall that we have proved (twice) that the solutions to this equation disperse (in dimensions $n \geq 2$), but we are yet to give a proof using energy methods. This is the goal of this section, which follows some breakthrough ideas of Klainerman. Of course, we mention that one of the important reasons that we prove dispersion using the energy method is that it is more robust and can be applied to nonlinear problems.

We begin with the following simple lemma:

Lemma 9.1. *If $\square\phi = 0$, then*

$$\square(\Gamma\phi) = 0,$$

where Γ is one of the vector fields $\Gamma \in \{\partial_t, \partial_i, \Omega_{ij} := x_i\partial_j - x_j\partial_i, S := t\partial_t + \sum_{i=1}^n x_i\partial_i, \Omega_{0i} := t\partial_i + x_i\partial_t\}$.

Proof. This is an easy calculation. In particular for $\Gamma \in \{\partial_t, \partial_i, x_i\partial_j - x_j\partial_i, t\partial_i + x_i\partial_t\}$,

$$\square(\Gamma\phi) = \Gamma(\square\phi)$$

and for $S = t\partial_t + \sum_{i=1}^n x_i\partial_i$, we have

$$\square(S\phi) = S(\square\phi) + 2\square\phi.$$

□

Definition 9.2. We call the set of admissible vector fields in Lemma 9.1 the **commuting vector fields**.

Obviously, we can then apply the result of the conservation of energy for each of these $\Gamma\phi$:

Corollary 9.3. *Given smooth and compactly supported initial data (ϕ_0, ϕ_1) to the linear wave equation, we have*

$$\sup_{t \in [0, \infty)} \sum_{|\alpha| \leq k} \|\partial\Gamma^\alpha\phi\|_{L^2(\mathbb{R}^n)}(t) = \sum_{|\alpha| \leq k} \|\partial\Gamma^\alpha\phi\|_{L^2(\mathbb{R}^n)}(0).$$

The key idea of Klainerman is that one can prove a weighted Sobolev embedding type result which takes advantage of the weights in the vector fields above that have good commutation properties with \square . More precisely, he proved the following global Sobolev inequality:

Theorem 9.4 (Klainerman Sobolev inequality). *There exists $C = C(n) > 0$ such that the following holds for all functions ϕ :*

$$\sup_x (1+t+r)^{\frac{n-1}{2}} (1+|t-r|)^{\frac{1}{2}} |\phi|(t, x) \leq C \sum_{|\alpha| \leq \frac{n+2}{2}} \|\Gamma^\alpha\phi\|_{L^2(\mathbb{R}^n)}(t).$$

Before we proceed to the proof, let us begin with some consequences of the theorem. The first easy corollary is the decay of the derivatives of ϕ :

Corollary 9.5. *Given data $(\phi_0, \phi_1) \in C_c^\infty \times C_c^\infty$, we have the following decay estimates*

$$\sup_x \sum_{|\alpha|=1} |\partial^\alpha\phi|(t, x) \leq \frac{C}{(1+t)^{\frac{n-1}{2}}} \sum_{|\alpha| \leq \frac{n+2}{2}} \|\partial\Gamma^\alpha\phi\|_{L^2(\mathbb{R}^n)}(0).$$

Of course this is the sharp decay rate for the derivatives that we obtained before using the explicit formula for the solutions. Notice that this method also has the additional advantage that it is now easy to see what norms of the initial data is the decay rate dependent on.

Recall that if $|x| \leq (1 - \epsilon)t$ or $|x| \geq (1 + \epsilon)t$ for some $\epsilon > 0$, we in fact have better decay (Corollary 2.5). Here, do not obtain a decay rate that is as good as that in Corollary 2.5, but we nonetheless see that there is an improvement:

Corollary 9.6. *Given data $(\phi_0, \phi_1) \in C_c^\infty \times C_c^\infty$, there exists $C = C(n, \epsilon) > 0$ such that the following decay estimate holds in $S_t := \{x : |x| < (1 - \epsilon)t \text{ or } |x| > (1 + \epsilon)t\}$:*

$$\sup_{x \in S_t} \sum_{|\alpha|=1} |\partial^\alpha \phi|(t, x) \leq \frac{C}{(1+t)^{\frac{n}{2}}} \sum_{|\alpha| \leq \frac{n+2}{2}} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^n)}(0).$$

Of course the natural question is what we can say about ϕ itself using this method. Notice that if the initial data are compactly supported, we can use finite speed of propagation and integrate in the $u = t - r$ direction to get the the following result:

Corollary 9.7. *Suppose the initial data (ϕ_0, ϕ_1) are compactly supported in $B(0, R)$. Then, there exists $C = C(n, R)$ such that*

$$\sup_x (1+t+r)^{\frac{n-1}{2}} (1+|t-r|)^{-\frac{1}{2}} |\phi| \leq C \sum_{|\alpha| \leq \frac{n+2}{2}} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^n)}(0).$$

Note that this decay rate is worse than the sharp rate for compactly supported data. In 3 + 1 dimensions, one can in fact get the sharp decay rate $|\phi| \leq \frac{C}{1+t}$ by not only introducing the commuting vector fields Γ 's but also introducing a different energy. See example sheet.

Moreover, we see that in fact some derivatives of ϕ decay better than the others! This fact is not so easy to see even with the formulas for explicit solutions. However, it is an easy consequence of the methods above. Before we state the result, let us begin by introducing some notations: Let $v = t + r$ and $u = t - r$, where $r = |x|$. Notice that we have

$$(\partial_t \phi)^2 + \sum_{i=1}^n (\partial_i \phi)^2 = (\partial_t \phi)^2 + (\partial_r \phi)^2 + |\nabla \phi|^2 = 2(\partial_v \phi)^2 + 2(\partial_u \phi)^2 + |\nabla \phi|^2.$$

Here, $|\nabla \phi|^2$ is the angular part of the derivatives, whose norm is defined by

$$|\nabla \phi|^2 := \frac{1}{2} \sum_{i,j=1}^3 \left(\frac{x_i}{r} \partial_j \phi - \frac{x_j}{r} \partial_i \phi \right)^2.$$

We finally define the ‘‘good derivatives’’ $\bar{\partial}$ to be either the ∂_v derivative or the angular derivatives, i.e.

$$|\bar{\partial} \phi|^2 := 2(\partial_v \phi)^2 + |\nabla \phi|^2.$$

Corollary 9.8. *Suppose the initial data (ϕ_0, ϕ_1) are compactly supported in $B(0, R)$. Then, there exists $C = C(n, R)$ such that*

$$(1+t+r)^{\frac{n+1}{2}} (1+|t-r|)^{-\frac{1}{2}} |\bar{\partial} \phi| \leq C \sum_{|\alpha| \leq \frac{n+4}{2}} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^n)}(0).$$

Proof. It suffices to establish the bound

$$(9.1) \quad |\bar{\partial} \phi| \leq \frac{C \sum_{|\alpha|=1} |\Gamma \phi|}{1+t+r}$$

so that we can apply Corollary 9.7. Clearly, we only need to consider the case $t + r > 1$ for in the case $t + r \leq 1$, the right hand side controls all derivatives. To conclude, we note that

$$\partial_v = \frac{S + \sum_{i=1}^n \frac{x_i}{r} \Omega_{0i}}{2(t+r)}, \quad \Omega_{ij} = \frac{1}{t} (x_i \Omega_{0j} - x_j \Omega_{0i}).$$

Also, by definition,

$$|\nabla \phi| \leq \frac{C}{r} \sum_{i,j=1}^n |\Omega_{ij} \phi|,$$

which implies also that

$$|\nabla\phi| \leq \frac{C}{t} \sum_{i=1}^n |\Omega_{0i}\phi|$$

by the above identity. The result follows. \square

Remark 9.9. In the physical 3 + 1 dimensions, note the following important consequence of Corollary 9.8: $\sup_x |\partial\phi|(t, x)$ is in general not integrable in time, while $\sup_x |\bar{\partial}\phi|(t, x)$ is!

We now turn to the proof of Theorem 9.4. We begin with two lemmas. First, we have

Lemma 9.10. *There exists $C = C(n, k) > 0$ such that*

$$\sum_{|\alpha|=k} |\partial^\alpha\phi| \leq \frac{C}{|t-r|^k} \sum_{|\alpha|\leq k} |\Gamma^\alpha\phi|.$$

Proof. This follows from apply iteratively the identities

$$\partial_i = \frac{-x_j\Omega_{ij} + t\Omega_{0i} - x_iS}{t^2 - r^2}$$

and

$$\partial_t = \frac{tS - x_i\Omega_{0i}}{t^2 - r^2}.$$

\square

The second lemma is a scale-invariant version of the Sobolev embedding theorem:

Lemma 9.11. *There exists $C = C(n, k)$ such that the following holds for all functions ϕ :*

$$(9.2) \quad \|\phi\|_{L^\infty} \leq C \|\phi\|_{L^2(\mathbb{R}^n)}^{\frac{2k-n}{2k}} \left(\sum_{|\alpha|=k} \|\partial^\alpha\phi\|_{L^2(\mathbb{R}^n)} \right)^{\frac{n}{2k}}.$$

Proof. We begin with the following

$$\|\phi\|_{L^\infty} \leq C \|\phi\|_{H^k(\mathbb{R}^n)} \leq C (\|\phi\|_{L^2(\mathbb{R}^n)} + \sum_{|\alpha|=k} \|\partial^\alpha\phi\|_{L^2(\mathbb{R}^n)})$$

for $k \in \mathbb{Z}$, $k > \frac{n}{2}$. Introduce now a scaling parameter λ and rescale ϕ by

$$\phi_\lambda(x) = \phi(\lambda x).$$

Notice that for every $m \in \mathbb{Z}$, we have

$$\sum_{|\alpha|=m} \|\phi_\lambda\|_{L^2(\mathbb{R}^n)} = \lambda^{m-\frac{n}{2}} \sum_{|\alpha|=m} \|\phi\|_{L^2(\mathbb{R}^n)}.$$

Therefore, by the Sobolev embedding theorem above, which holds for all ϕ_λ , we have a family of estimates

$$\|\phi\|_{L^\infty} \leq C \left(\lambda^{-\frac{n}{2}} \|\phi\|_{L^2(\mathbb{R}^n)} + \lambda^{k-\frac{n}{2}} \sum_{|\alpha|=k} \|\partial^\alpha\phi\|_{L^2(\mathbb{R}^n)} \right).$$

Choosing $\lambda = \left(\sum_{|\alpha|=k} \|\partial^\alpha\phi\|_{L^2(\mathbb{R}^n)} \right)^{-\frac{1}{k}} \left(\|\phi\|_{L^2(\mathbb{R}^n)} \right)^{\frac{1}{k}}$, we get the desired conclusion. \square

We are now ready to prove Theorem 9.4:

Proof. (1) We first consider the regions $\{r \leq \frac{t}{2}\}$ and $\{r \geq 2t\}$. By standard Sobolev embedding theorem, we can clearly assume that $|t+r| \geq 1$. Now, we can use Lemma 9.10 to exchange ∂ with Γ while gaining a weight in $|t-r|^{-1}$. In order to ensure that the weight inside the integral is comparable to that outside the integral, we now introduce appropriate cutoff functions and use the estimate above. First, let $\chi(\frac{r}{t})$ be a cutoff function such that

$$\chi(x) = \begin{cases} 1, & \text{if } x \leq \frac{1}{2} \\ 0, & \text{if } x \geq \frac{3}{4}. \end{cases}$$

Using Lemma 9.10, we have

$$\begin{aligned} \sum_{|\alpha|=k} \|\partial^\alpha(\chi\phi)\|_{L^2(\mathbb{R}^n)} &\leq C \sum_{|\alpha|\leq k} t^{-k+|\alpha|} \left(\int_{\{r\leq\frac{3t}{4}\}} |\partial^\beta\phi|^2(x) dx \right)^{\frac{1}{2}} \\ &\leq C \sum_{|\alpha|\leq k} t^{-k} \left(\int_{\{r\leq\frac{3t}{4}\}} |\Gamma^\beta\phi|^2(x) dx \right)^{\frac{1}{2}} \end{aligned}$$

Returning to (9.2), we get the desired result when $r \leq \frac{t}{2}$. The case $r \geq 2t$ can be treated similarly by introducing an appropriate cutoff and using (9.2).

- (2) With the result in part (1), it remains to consider the region $\{\frac{t}{2} \leq r \leq 2t\}$. We can assume that $|t-r| \geq 1$. Let us first explain our strategy in a heuristic manner. As we saw above in step (1), what we basically did was to gain a weight of $|t-r|^{\frac{1}{2}}$ for the Sobolev embedding in each direction. To sharpen this when $|t-r|$ is small, note that we can separate the angular directions and the radial direction and for each of the angular directions, we in fact gain $r^{\frac{1}{2}}$. Now, in the region that we are interested in, i.e., $\{\frac{t}{2} \leq r \leq 2t\}$, t is comparable of r and therefore the gain in r is equivalent to t .

We now turn to the details. We write $x = (r, \vartheta)$, where $\vartheta \in \mathbb{S}^{n-1}$. Denote also by $d\sigma_\vartheta$ the standard measure on the sphere \mathbb{S}^{n-1} of radius 1. Notice that the volume form on \mathbb{R}^n can be written as $r^{n-1} dr d\sigma_\vartheta$. Introduce a cutoff function $\chi(\frac{r}{t})$ such that

$$\chi(x) = \begin{cases} 1, & \text{if } \frac{1}{2} \leq x \leq 2 \\ 0, & \text{if } x < \frac{1}{4} \text{ or } x > 4. \end{cases}$$

where χ is smooth, compactly supported and $0 \leq \chi \leq 1$ everywhere. Obviously it suffices to control $\chi(\frac{r}{t})\phi$. First, we have the one dimensional inequality

$$|\chi(\frac{r}{t})\psi(r, \vartheta)| \leq C \min\left\{ \int_{\frac{t}{4}}^r |\partial_r(\chi\psi)|(r', \vartheta) dr', \int_r^{4t} |\partial_r(\chi\psi)|(r', \vartheta) dr' \right\}$$

for any smooth function ψ , using the support properties of χ . On the other hand, we have the Sobolev embedding theorem on the sphere, i.e., there exists $C = C(n) > 0$ such that

$$|\phi| \leq C \left(\sum_{|\alpha|\leq\frac{n}{2}} \int_{\mathbb{S}^{n-1}} |\Omega_{ij}^\alpha \phi|^2(\vartheta) d\sigma_\vartheta \right)^{\frac{1}{2}}.$$

First, assume that $r \leq t$.

$$\begin{aligned} |\chi\phi|(t, x) &\leq C \sum_{|\alpha|\leq\frac{n}{2}} \left(\int_{\mathbb{S}^{n-1}} |\chi|^2 |\Omega_{ij}^\alpha \phi|^2(r, \vartheta) d\sigma_\vartheta \right)^{\frac{1}{2}} \\ &\leq C \sum_{|\alpha|\leq\frac{n}{2}} \left(\int_{\mathbb{S}^{n-1}} \left(\int_{\frac{t}{4}}^r |\partial_r(\chi\Omega_{ij}^\alpha \phi)|(r', \vartheta) dr' \right)^2 d\sigma_\vartheta \right)^{\frac{1}{2}} \\ &\leq C \sum_{|\alpha|\leq\frac{n}{2}} \sum_{|\beta|=1} \left(\int_{\mathbb{S}^{n-1}} \left(\int_{\frac{t}{4}}^r (1+|t-r'|)^{-1} |\Gamma^\beta(\chi\Omega_{ij}^\alpha \phi)|(r', \vartheta) dr' \right)^2 d\sigma_\vartheta \right)^{\frac{1}{2}} \\ &\leq \frac{C}{|t+r|^{\frac{n-1}{2}} |t-r|^{\frac{1}{2}}} \sum_{|\alpha|\leq\frac{n}{2}} \sum_{|\beta|=1} \left(\int_{\mathbb{S}^{n-1}} \int_{\frac{t}{4}}^r |\Gamma^\beta(\chi\Omega_{ij}^\alpha \phi)|^2(r', \vartheta) (r')^{n-1} dr' d\sigma_\vartheta \right)^{\frac{1}{2}} \\ &\leq \frac{C}{|t+r|^{\frac{n-1}{2}} |t-r|^{\frac{1}{2}}} \sum_{|\alpha|\leq\frac{n}{2}+1} \left(\int_{\mathbb{R}^n} |\Gamma^\alpha \phi|^2(r', \vartheta) (r')^{n-1} dr' d\sigma_\vartheta \right)^{\frac{1}{2}}, \end{aligned}$$

where in the last line we have used that for $\frac{t}{4} \leq r \leq t$, we have $\sum_{|\alpha|=1} |\Gamma^\alpha \chi| \leq C$.

The case $r > t$ can be treated similarly by integrating from $r = 4t$ instead of $r = \frac{t}{4}$:

$$\begin{aligned} |\chi\phi|(t, x) &\leq C \sum_{|\alpha| \leq \frac{n}{2}} \left(\int_{\mathbb{S}^{n-1}} |\chi|^2 |\Omega_{ij}^\alpha \phi|^2(r, \vartheta) d\sigma_\vartheta \right)^{\frac{1}{2}} \\ &\leq C \sum_{|\alpha| \leq \frac{n}{2}} \left(\int_{\mathbb{S}^{n-1}} \left(\int_r^{4t} |\partial_r(\chi \Omega_{ij}^\alpha \phi)|(r', \vartheta) dr' \right)^2 d\sigma_\vartheta \right)^{\frac{1}{2}} \\ &\leq \frac{C}{|t+r|^{\frac{n-1}{2}} |t-r|^{\frac{1}{2}}} \sum_{|\alpha| \leq \frac{n}{2}+1} \left(\int_{\mathbb{R}^n} |\Gamma^\alpha \phi|^2(r', \vartheta) (r')^{n-1} dr' d\sigma_\vartheta \right)^{\frac{1}{2}}. \end{aligned}$$

This concludes the proof of the theorem. □

10. SMALL DATA GLOBAL REGULARITY OF SOLUTIONS

In the rest of the course, we will be concerned with small data solutions to nonlinear equations. As we mentioned in Section 7, very little is known about the global behaviour of solutions to supercritical equations with general data. On the other hand, much more can be said about these equations when restricted to small initial data.

Roughly speaking, this is because when the initial data are small, then the nonlinear terms are even smaller and thus the linear terms dominate so that we can control the nonlinear terms. However, recall that when we estimate the error terms using energy estimates, they have to be integrated in time. In order to actually control these terms globally in time, we must therefore show that the same time that the solution decays sufficiently fast and the error terms are integrable. It turns out that the methods introduced in Section 9 can easily be adapted for nonlinear equations and are very well-suited for the purpose of showing the decay of the solutions.

We begin with a simple example illustrating the method.

Theorem 10.1. *Let $k \geq 6$. Consider the wave map equation*

$$\square\phi = -\phi(\partial_t\phi^t\partial_t\phi - \sum_{i=1}^4 \partial_i\phi^t\partial_i\phi).$$

in $\mathbb{R} \times \mathbb{R}^4$ with initial data⁷

$$(\phi_0, \phi_1)|_{\{t=0\}} \in C_c^\infty(B(0, R)) \times C_c^\infty(B(0, R))$$

such that

$$\sum_{|\alpha| \leq k} \|\partial\partial^\alpha\phi_0\|_{L^2(\mathbb{R}^4)} + \|\partial^\alpha\phi_1\|_{L^2(\mathbb{R}^4)} < \epsilon.$$

Then for every $R > 0$, there exists $\epsilon_0 = \epsilon_0(R) > 0$ sufficiently small such that if $\epsilon \leq \epsilon_0$, the unique solution remains smooth for all time.

Proof. We prove this using the bootstrap method. We will assume that some weighted energy is bounded by some constant. Then by Klainerman-Sobolev inequality, we can show that the solution decays. Because the initial data are small, this allows us to prove that the weighted energy is bounded by some better constant. Thus by continuity, we conclude that the weighted energy cannot grow to infinity in any finite time interval and hence using the local existence theorem, the solution exists for all time.

We now turn to the details. Assume that

$$(10.1) \quad \sup_{t \in [0, T]} \sum_{|\alpha| \leq k} \|\partial\Gamma^\alpha\phi\|_{L^2(\mathbb{R}^4)}(t) \leq \epsilon^{\frac{3}{4}}$$

for $T < T_*$. Notice that ϵ sufficiently small, we have $\epsilon \ll \epsilon^{\frac{3}{4}}$ and hence this holds initially.

⁷Notice that by compactly supported, we mean that the map coincide with the constant map outside a ball. Moreover, as always, when we prescribe initial data for the wave map problem, we require $|\phi_0|^2 = 1$ and $\phi_1^t\phi_0 = 0$.

By Klainerman-Sobolev inequality (and the fact that $k \geq 6$), we therefore have that for some constant $C > 0$,

$$(10.2) \quad \sum_{|\alpha| \leq \frac{k}{2}} \|(1+t+|x|)^{\frac{3}{2}} \partial \Gamma^\alpha \phi\|_{L^\infty(\mathbb{R}^4)}(t) \leq C\epsilon^{\frac{3}{4}}.$$

This further implies that we have

$$(10.3) \quad \sum_{1 \leq |\alpha| \leq \frac{k}{2} + 1} \|(1+t+|x|)^{\frac{1}{2}} \Gamma^\alpha \phi\|_{L^\infty(\mathbb{R}^4)}(t) \leq C\epsilon^{\frac{3}{4}}.$$

This is because we can trivially bound

$$\sum_{1 \leq |\alpha| \leq \frac{k}{2} + 1} |\Gamma^\alpha \phi| \leq C(1+t+|x|) \sum_{|\alpha| \leq \frac{k}{2}} |\partial \Gamma^\alpha \phi|.$$

We now turn to the energy estimates. For some constant depending on k (but independent of T), we have

$$\begin{aligned} & \sup_{t \in [0, T]} \sum_{|\alpha| \leq k} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^4)}(t) \\ & \leq C(k) \left(\sum_{|\alpha| \leq k} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^4)}(0) + \int_0^T \sum_{|\alpha| \leq k} \|\Gamma^\alpha \left(\phi(\partial_t \phi^t \partial_t \phi - \sum_{i=1}^n \partial_i \phi^t \partial_i \phi) \right)\|_{L^2(\mathbb{R}^4)}(t) dt \right). \end{aligned}$$

Notice that

$$\begin{aligned} & \sum_{|\alpha| \leq k} \left| \Gamma^\alpha \left(\phi(\partial_t \phi^t \partial_t \phi - \sum_{i=1}^n \partial_i \phi^t \partial_i \phi) \right) \right| \\ & \leq C \left(\sum_{|\alpha_1| + |\alpha_2| \leq k} |\partial \Gamma^{\alpha_1} \phi| |\partial \Gamma^{\alpha_2} \phi| + \sum_{|\alpha_1| + |\alpha_2| + |\alpha_3| \leq k, |\alpha_1| \geq 1} |\Gamma^{\alpha_1} \phi| |\partial \Gamma^{\alpha_2} \phi| |\partial \Gamma^{\alpha_3} \phi| \right). \end{aligned}$$

We estimate the first term. Clearly, either $|\alpha_1| \leq \frac{k}{2}$ or $|\alpha_2| \leq \frac{k}{2}$. Without loss of generality, we assume $|\alpha_1| \leq \frac{k}{2}$. We can therefore apply (10.2). More precisely,

$$\begin{aligned} & \sum_{|\alpha_1| + |\alpha_2| \leq k} \int_0^T \|\partial \Gamma^{\alpha_1} \phi\| \|\partial \Gamma^{\alpha_2} \phi\|_{L^2(\mathbb{R}^4)}(t) dt \\ & \leq C \int_0^T \left(\sum_{|\alpha_1| \leq \frac{k}{2}} \|\partial \Gamma^{\alpha_1} \phi\|_{L^\infty(\mathbb{R}^4)} \right) \left(\sum_{|\alpha_2| \leq k} \|\partial \Gamma^{\alpha_2} \phi\|_{L^2(\mathbb{R}^4)}(t) \right) dt \\ & \leq C\epsilon^{\frac{3}{4}} \int_0^T \frac{(\sum_{|\alpha_2| \leq k} \|\partial \Gamma^{\alpha_2} \phi\|_{L^2(\mathbb{R}^4)})}{(1+t)^{\frac{3}{2}}} dt \leq C\epsilon^{\frac{3}{2}}, \end{aligned}$$

where we have used (10.1) in the last step. The most important point is that $\frac{1}{(1+t)^{\frac{3}{2}}}$ is integrable in time and therefore this term can be bounded independent of T ! We now look at the cubic term. At least two of $|\alpha_1|$, $|\alpha_2|$ and $|\alpha_3|$ is $\leq \frac{k}{2}$. There are two distinct cases: $|\alpha_1|, |\alpha_2| \leq \frac{k}{2}$ and $|\alpha_2|, |\alpha_3| \leq \frac{k}{2}$. More precisely, we

have

$$\begin{aligned}
& \sum_{|\alpha_1|+|\alpha_2|\leq k} \int_0^T \|\Gamma^{\alpha_1}\phi\|\|\partial\Gamma^{\alpha_2}\phi\|\|\partial\Gamma^{\alpha_3}\phi\|_{L^2(\mathbb{R}^4)}(t)dt \\
& \leq C \int_0^T \left(\sum_{1\leq|\alpha_1|\leq\frac{k}{2}} \|\Gamma^{\alpha_1}\phi\|_{L^\infty(\mathbb{R}^4)} \right) \left(\sum_{|\alpha_2|\leq\frac{k}{2}} \|\partial\Gamma^{\alpha_2}\phi\|_{L^\infty(\mathbb{R}^4)} \right) \left(\sum_{|\alpha_3|\leq k} \|\partial\Gamma^{\alpha_3}\phi\|_{L^2(\mathbb{R}^4)} \right) (t) dt \\
& \quad + C \int_0^T \left(\sum_{1\leq|\alpha_1|\leq k} \|\Gamma^{\alpha_1}\phi\|_{L^\infty(\mathbb{R}^4)} \right) \left(\sum_{|\alpha_2|\leq\frac{k}{2}} \|\partial\Gamma^{\alpha_2}\phi\|_{L^\infty(\mathbb{R}^4)} \right) \left(\sum_{|\alpha_3|\leq\frac{k}{2}} \|\partial\Gamma^{\alpha_3}\phi\|_{L^2(\mathbb{R}^4)} \right) (t) dt \\
& \leq C \int_0^T \left(\sum_{1\leq|\alpha_1|\leq\frac{k}{2}} \|\Gamma^{\alpha_1}\phi\|_{L^\infty(\mathbb{R}^4)} \right) \left(\sum_{|\alpha_2|\leq\frac{k}{2}} \|\partial\Gamma^{\alpha_2}\phi\|_{L^\infty(\mathbb{R}^4)} \right) \left(\sum_{|\alpha_3|\leq k} \|\partial\Gamma^{\alpha_3}\phi\|_{L^2(\mathbb{R}^4)} \right) (t) dt \\
& \quad + C \int_0^T (1+t) \left(\sum_{|\alpha_1|\leq k-1} \|\partial\Gamma^{\alpha_1}\phi\|_{L^\infty(\mathbb{R}^4)} \right) \left(\sum_{|\alpha_2|\leq\frac{k}{2}} \|\partial\Gamma^{\alpha_2}\phi\|_{L^\infty(\mathbb{R}^4)} \right) \left(\sum_{|\alpha_3|\leq\frac{k}{2}} \|\partial\Gamma^{\alpha_3}\phi\|_{L^2(\mathbb{R}^4)} \right) (t) dt \\
& \leq C\epsilon^{\frac{3}{2}} \int_0^T \frac{(\sum_{|\alpha|\leq k} \|\partial\Gamma^\alpha\phi\|_{L^2(\mathbb{R}^4)})}{(1+t)^2} dt \leq C\epsilon^{\frac{9}{4}}.
\end{aligned}$$

Combining the estimates above, we have thus obtained

$$\begin{aligned}
& \sup_{t\in[0,T]} \sum_{|\alpha|\leq k} \|\partial\Gamma^\alpha\phi\|_{L^2(\mathbb{R}^4)}(t) \\
& \leq C(k) \left(\sum_{|\alpha|\leq k} \|\partial\Gamma^\alpha\phi\|_{L^2(\mathbb{R}^4)}(0) + \epsilon^{\frac{3}{2}} \right) \leq C\epsilon.
\end{aligned}$$

Since $C\epsilon \leq \frac{\epsilon^{\frac{3}{2}}}{2}$, we have thus improved the constant in (10.1) and concluded the argument. \square

The above theorem is of course for the wave map equations in 4 + 1 dimensions. However, as one can see from the proof, the method can be quite generally applied in a neighbourhood of the zero solution for a large class nonlinear wave equation with quadratic nonlinear in the derivatives of ϕ . On the other hand, it is important to note that the proof fails in 3 + 1 dimensions! Indeed, that case requires a more delicate analysis and will be the subject we turn to in the next section.

11. THE NULL CONDITION

As we have seen in the previous section, in order to prove the small data global regularity result for the wave map equations in 4 + 1 dimensions, we have crucially used the fact that $\int_0^\infty \frac{dt}{(1+t)^{\frac{3}{2}}} < \infty$. The same proof however fails in 3+1 dimensions since in that case, we only have a decay rate of $\frac{1}{1+t}$ and $\int_0^\infty \frac{dt}{(1+t)} = \infty$! Nevertheless, the same result is in fact also true, but requires a more delicate argument. There are two main observations, the first is in the following lemma:

Lemma 11.1. *The following identity holds*

$$\partial_t\phi\partial_t\psi - \sum_{i=1}^3 \partial_i\phi\partial_i\psi = 2\partial_u\phi\partial_v\psi + 2\partial_v\phi\partial_u\psi - \nabla\phi \cdot \nabla\psi,$$

where $\nabla\phi$ is defined as an n -dimensional vector with components given by

$$\nabla_i\phi := \partial_i\phi - \frac{x_i}{r}\partial_r\phi = \sum_{j=1}^n \frac{x_j}{r^2}\Omega_{ij}\phi.$$

The dot product \cdot here is simply the standard dot product in \mathbb{R}^n . Note that we have

$$\nabla\phi \cdot \nabla\psi := \frac{1}{2r^2} \sum_{i,j=1}^3 \Omega_{ij}\phi\Omega_{ij}\psi.$$

Proof. It is easy to check that

$$\sum_{i=1}^3 \partial_i \phi \partial_i \psi = \partial_r \phi \partial_r \psi + \frac{1}{2r^2} \sum_{i,j=1}^3 \Omega_{ij} \phi \Omega_{ij} \psi$$

and

$$\partial_t \phi \partial_t \psi - \partial_r \phi \partial_r \psi = \frac{1}{2} (\partial_t \phi + \partial_r \phi) (\partial_t \psi - \partial_r \psi) + \frac{1}{2} (\partial_t \phi - \partial_r \phi) (\partial_t \psi + \partial_r \psi).$$

□

In other words, in any quadratic terms, at least one of the derivatives is a $\bar{\partial}$ derivative, where as before $\bar{\partial} \in \{\partial_v, \nabla\}$. As we have seen previously (see Corollary 9.8), in fact $|\bar{\partial}\phi|$ decays better than $|\partial\phi|$. Most importantly, the decay is integrable in time! This is good news, but before we declare victory, remember that we need to put one of the two factors in the quadratic term in L^2 and the other in L^∞ . In particular, we need to show that not only is $|\bar{\partial}\phi|$ better in L^∞ norm, it is also better in some L^2 sense. This is provided by the following proposition:

Proposition 11.2. *Let*

$$\square\phi = F.$$

For every $\delta > 0$, there exists $C = C(\delta) > 0$ such that

$$\left(\int_0^T \int_{\mathbb{R}^3} \frac{|\bar{\partial}\phi|^2}{(1+|t-|x||)^{1+\delta}} dx dt \right)^{\frac{1}{2}} \leq C \left(\|(\phi_0, \phi_1)\|_{\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)} + \int_0^T \|F\|_{L^2(\mathbb{R}^3)}(t) dt \right).$$

Proof. Let $w : \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 function to be determined.

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^3} w(t-|x|) \partial_t \phi F dx dt \\ &= \int_0^T \int_{\mathbb{R}^3} w(t-|x|) \partial_t \phi \square\phi dx dt \\ &= -\frac{1}{2} \int_{\mathbb{R}^3} w(T-|x|) \left((\partial_t \phi)^2 + \sum_{i=1}^3 (\partial_i \phi)^2 \right) dx(T) + \frac{1}{2} \int_{\mathbb{R}^3} w(-|x|) \left((\partial_t \phi)^2 + \sum_{i=1}^3 (\partial_i \phi)^2 \right) dx(0) \\ & \quad + \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} w'(t-|x|) \left((\partial_t \phi)^2 + \sum_{i=1}^3 \frac{2x_i}{r} \partial_t \phi \partial_i \phi + \sum_{i=1}^3 (\partial_i \phi)^2 \right) dx dt. \end{aligned}$$

Observe now that

$$\begin{aligned} & (\partial_t \phi)^2 + \sum_{i=1}^3 \frac{2x_i}{r} \partial_t \phi \partial_i \phi + \sum_{i=1}^3 (\partial_i \phi)^2 \\ &= (\partial_t \phi)^2 + 2\partial_t \phi \partial_r \phi + (\partial_r \phi)^2 + |\nabla\phi|^2 = 4(\partial_v \phi)^2 + |\nabla\phi|^2. \end{aligned}$$

Now, we choose w such that it is bounded and decreasing and satisfying $-w'(s) \geq c(1+|s|)^{-1-\delta}$ (where c is a constant which can depend on δ). This can be achieved for instance by

$$w(s) = \begin{cases} (1+s)^{-\delta} & \text{if } s \geq 0, \\ 1 + \delta \int_s^0 \frac{dq}{(1+|q|)^{-1-\delta}} & \text{if } s < 0. \end{cases}$$

Combining all these imply that

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^3} \frac{|\bar{\partial}\phi|^2}{(1+|t-|x||)^{1+\delta}} dx dt \\ & \leq C \left(\|(\phi_0, \phi_1)\|_{\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)}^2 + \int_0^T \left(\sup_{t \in [0, T]} \|\partial\phi\|_{L^2(\mathbb{R}^3)} \right) \|F\|_{L^2(\mathbb{R}^3)}(t) dt \right). \end{aligned}$$

and the conclusion follows from standard energy estimates (see Theorem 4.6). □

Let's look at what we have achieved. We now have an L^2 estimate which is integrated in time but contains a weight. Notice that the key point is that the weight is in $|t - |x||$ - indeed, the corresponding estimate with $t + |x|$ weight for

$$\int_0^T \int_{\mathbb{R}^3} \frac{|\partial\phi|^2}{(1+t+|x|)^{1+\delta}} dx dt$$

holds trivially since $\frac{1}{(1+t)^{1+\delta}}$ is integrable in t . The point is now that we also have a pointwise decay $|\partial\phi| \leq \frac{C}{(1+t+|x|)(1+|t-|x||)^{\frac{1}{2}}}$. The above estimate, which degenerates when $|t - |x||$ is large, allows us to exploit this extra decay in $|t - |x||$, which was previously not used in the 4 + 1 dimensional case.

Now we have good estimates in both L^∞ and L^2 for the good estimate $\bar{\partial}$. Before we state and prove the main theorem, the remaining thing to make sure is that after we differentiate the nonlinear term, we still preserve the structure that at least one of the factors has a good derivative. To this end, it is convenient to first introduce a larger class of bilinear forms which have the property that at least one of the factors has a good derivative.

Definition 11.3. Let $q^{\alpha\beta}$ be constants. We say that $Q(\phi, \psi) = q^{\alpha\beta} \partial_\alpha \phi \partial_\beta \psi$ satisfies the **classical null condition** if

$$q^{\alpha\beta} \xi_\alpha \xi_\beta = 0, \text{ whenever } m^{\alpha\beta} \xi_\alpha \xi_\beta = 0.$$

When Q satisfies the classical null condition, we also say that Q is a **classical null form**.

It is easy to classify all classical null forms and show that it has the desired property:

Lemma 11.4. *Let Q be a classical null form. Then it is a linear combination of $Q_0(\phi, \psi) := m^{\alpha\beta} \partial_\alpha \phi \partial_\beta \psi$ and $Q_{\mu\nu}(\phi, \psi) := \partial_\mu \phi \partial_\nu \psi - \partial_\nu \phi \partial_\mu \psi$. Moreover, we have*

$$|Q(\phi, \psi)| \leq C(|\partial\phi| |\bar{\partial}\psi| + |\bar{\partial}\phi| |\partial\psi|).$$

Proof. Clearly $Q_{\mu\nu}$ span the space of anti-symmetric bilinear forms (which has dimension $\frac{(n+1)!}{2!(n-1)!}$), all of which are null forms. It remains to show that all symmetric null forms are multiples of Q_0 . This is obvious since $q^{\alpha\beta} \xi_\alpha \xi_\beta$ determines a homogeneous polynomial in ξ of degree 2 and has the zero set that coincides with the polynomial $\xi_0^2 - \sum_{i=1}^n \xi_i^2$. The final statement can be checked directly for each of Q_0 and $Q_{\mu\nu}$. Q_0 is checked in Lemma 11.1 and we leave it as an exercise for the readers to show that $Q_{\mu\nu}$ also verifies the desired property. \square

As mentioned above, we then show that the null structure is ‘‘preserved’’ after differentiating by Γ :

Lemma 11.5. *Given a classical null form Q . Then for every commuting vector field $\Gamma \in \{\partial_t, \partial_i, \Omega_{ij}, \Omega_{0i}, S\}$, we have*

$$\Gamma(Q(\phi, \psi)) = Q(\Gamma\phi, \psi) + Q(\phi, \Gamma\psi) + \tilde{Q}(\phi, \psi),$$

for some classical null form \tilde{Q} .

Proof. We check this for Q_0 . We write⁸ $\Gamma = \Gamma^\alpha \partial_\alpha$ to get

$$\begin{aligned} & \Gamma(m^{\alpha\beta} \partial_\alpha \phi \partial_\beta \psi) \\ &= (m^{\alpha\beta} \partial_\alpha (\Gamma\phi) \partial_\beta \psi) + (m^{\alpha\beta} \partial_\alpha \phi \partial_\beta (\Gamma\psi)) - (m^{\alpha\beta} (\partial_\alpha \Gamma^\sigma) \partial_\sigma \phi \partial_\beta \psi) - (m^{\alpha\beta} (\partial_\beta \Gamma^\sigma) \partial_\alpha \phi \partial_\sigma \psi) \\ &= (m^{\alpha\beta} \partial_\alpha (\Gamma\phi) \partial_\beta \psi) + (m^{\alpha\beta} \partial_\alpha \phi \partial_\beta (\Gamma\psi)) - m^{\alpha\beta} (\partial_\alpha \Gamma^\sigma) (\partial_\sigma \phi \partial_\beta \psi + \partial_\beta \phi \partial_\sigma \psi). \end{aligned}$$

Now it is easy to check that for any commuting vector fields Γ , the last term can be written as a linear combination of null forms. $Q_{\mu\nu}$ can be checked in a similar manner and we leave it as an exercise. \square

We are now ready to state and prove the main theorem of this section:

Theorem 11.6. *Consider the wave map equation*

$$\square\phi = -\phi(\partial_t \phi^t \partial_t \phi - \sum_{i=1}^3 \partial_i \phi^t \partial_i \phi).$$

⁸Not to be confused with our usual notation that α is a multi-index!

in $\mathbb{R} \times \mathbb{R}^3$ with initial data⁹

$$(\phi_0, \phi_1) \upharpoonright_{\{t=0\}} \in C_c^\infty(B(0, R)) \times C_c^\infty(B(0, R))$$

such that

$$\sum_{|\alpha| \leq 5} \|\partial \partial^\alpha \phi_0\|_{L^2(\mathbb{R}^3)} + \|\partial^\alpha \phi_1\|_{L^2(\mathbb{R}^3)} < \epsilon.$$

Then for every $R > 0$, there exists $\epsilon_0 = \epsilon_0(R) > 0$ sufficiently small such that if $\epsilon \leq \epsilon_0$, the unique solution remains smooth for all time.

Proof. We will use a bootstrap argument as in the $(4+1)$ -dimensional case. Fix $\delta \in (0, 1)$. The constant C in the proof of the theorem is allowed to depend on δ and R (but is independent of T and ϵ). Assume that

$$(11.1) \quad \sup_{t \in [0, T]} \sum_{|\alpha| \leq 5} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^3)} + \sum_{|\alpha| \leq 5} \left(\int_0^T \int_{\mathbb{R}^3} \frac{|\bar{\partial} \Gamma^\alpha \phi|^2}{(1 + |t - |x||)^{1+\delta}} dx dt \right)^{\frac{1}{2}} \leq \epsilon^{\frac{3}{4}}$$

for all T such that the solution remains regular and we will show that this bound in fact holds with a better constant. By Theorem 9.4, we have

$$(11.2) \quad \sup_{t \in [0, T]} \sum_{|\alpha| \leq 3} \|(1+t)(1+|t-|x|)^{\frac{1}{2}} \partial \Gamma^\alpha \phi\|_{L^\infty(\mathbb{R}^3)} \leq C \epsilon^{\frac{3}{4}}.$$

Using (9.1) together with (11.2), we get

$$(11.3) \quad \sup_{t \in [0, T]} \sum_{|\alpha| \leq 2} (1+t)^{\frac{3}{2}} \|\bar{\partial} \Gamma^\alpha \phi\|_{L^\infty(\mathbb{R}^3)} \leq C \epsilon^{\frac{3}{4}}.$$

(11.2) also implies that¹⁰

$$(11.4) \quad \sup_{t \in [0, T]} \sum_{|\alpha| \leq 3} \|\Gamma^\alpha \phi\|_{L^\infty(\mathbb{R}^3)} \leq C \epsilon^{\frac{3}{4}}.$$

We now use this pointwise decay bound to control the energy. By Theorem 4.6 and Proposition 11.2, we have

$$(11.5) \quad \begin{aligned} & \sup_{t \in [0, T]} \sum_{|\alpha| \leq 5} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^3)}(t) + \left(\int_0^T \int_{\mathbb{R}^3} \frac{|\bar{\partial} \phi|^2}{(1 + |t - |x||)^{1+\delta}} dx dt \right)^{\frac{1}{2}} \\ & \leq C \left(\epsilon + \int_0^T \sum_{|\alpha| \leq 5} \|\partial \Gamma^\alpha \left(\phi(\partial_t \phi^t \partial_t \phi - \sum_{i=1}^3 \partial_i \phi^t \partial_i \phi) \right)\|_{L^2(\mathbb{R}^3)}(t) dt \right). \end{aligned}$$

The error term can be bounded as follows:

$$\begin{aligned} & \int_0^T \sum_{|\alpha| \leq 5} \|\partial \Gamma^\alpha \left(\phi(\partial_t \phi^t \partial_t \phi - \sum_{i=1}^3 \partial_i \phi^t \partial_i \phi) \right)\|_{L^2(\mathbb{R}^3)}(t) dt \\ & \leq \int_0^T \left(\sum_{|\alpha| \leq 5} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^3)} \right) \left(\sum_{|\beta| \leq 2} \|\bar{\partial} \Gamma^\beta \phi\|_{L^\infty(\mathbb{R}^3)} \right) (t) dt \\ & \quad + \int_0^T \left(\sum_{|\alpha| \leq 5, |\beta| \leq 2} \|\bar{\partial} \Gamma^\alpha \phi \partial \Gamma^\beta \phi\|_{L^2(\mathbb{R}^3)} \right) (t) dt \\ & \quad + \int_0^T \left(\sum_{1 \leq |\alpha| \leq 5} \|\Gamma^\alpha \phi\|_{L^2(\mathbb{R}^3)} \right) \left(\sum_{|\beta| \leq 2} \|\partial \Gamma^\beta \phi\|_{L^\infty(\mathbb{R}^3)} \right) \left(\sum_{|\gamma| \leq 2} \|\bar{\partial} \Gamma^\gamma \phi\|_{L^\infty(\mathbb{R}^3)} \right) (t) dt. \end{aligned}$$

⁹As before, by compactly supported, we mean that the map coincide with the constant map outside a ball. Moreover, as always, when we prescribe initial data for the wave map problem, we require $|\phi_0|^2 = 1$ and $\phi_1^t \phi_0 = 0$.

¹⁰In fact we even have decay for this term, but we will not need to use that.

Notice that in the above, we have used (11.4). We now control each term

$$\int_0^T \left(\sum_{|\alpha| \leq 5} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^3)} \right) \left(\sum_{|\beta| \leq 2} \|\bar{\partial} \Gamma^\beta \phi\|_{L^\infty(\mathbb{R}^3)} \right) (t) dt \leq C \int_0^T \frac{\epsilon^{\frac{3}{2}} dt}{(1+t)^{\frac{3}{2}}} \leq C \epsilon^{\frac{3}{2}}.$$

We now control the second term. Here, we use the estimate in the second term of (11.1), which allows us to exploit the good derivative in L^2 .

$$\begin{aligned} & \int_0^T \left(\sum_{|\alpha| \leq 5, |\beta| \leq 2} \|\bar{\partial} \Gamma^\alpha \phi \partial \Gamma^\beta \phi\|_{L^2(\mathbb{R}^3)} \right) (t) dt \\ & \leq C \sum_{|\alpha| \leq 5, |\beta| \leq 2} \int_0^T \left(\int_{\mathbb{R}^3} |\bar{\partial} \Gamma^\alpha \phi \partial \Gamma^\beta \phi|^2 dx \right)^{\frac{1}{2}} dt \\ & \leq C \sum_{|\alpha| \leq 5, |\beta| \leq 2} \int_0^T \left(\int_{\mathbb{R}^3} \frac{|\bar{\partial} \Gamma^\alpha \phi|^2}{(1+|t-|x||)^{1+\delta}} dx \right)^{\frac{1}{2}} \left(\sup_x \left((1+|t-|x|)^{\frac{1+\delta}{2}} |\partial \Gamma^\beta \phi|(t, x) \right) \right) dt \\ & \leq C \left(\sum_{|\alpha| \leq 5} \int_0^T \int_{\mathbb{R}^3} \frac{|\bar{\partial} \Gamma^\alpha \phi|^2}{(1+|t-|x||)^{1+\delta}} dx dt \right)^{\frac{1}{2}} \left(\sum_{|\beta| \leq 2} \left(\int_0^T \left(\sup_x \left((1+|t-|x|)^{\frac{1+\delta}{2}} |\partial \Gamma^\beta \phi|(t, x) \right) \right)^2 dt \right)^{\frac{1}{2}} \right) \\ & \leq C \times \epsilon^{\frac{3}{4}} \times \left(\int_0^T \frac{\epsilon^{\frac{3}{2}}}{(1+t)^{2-\delta}} dt \right)^{\frac{1}{2}} \leq C \epsilon^{\frac{3}{2}}, \end{aligned}$$

for $\delta \in (0, 1)$. Finally, the last term can be controlled by noticing that

$$\sum_{1 \leq |\alpha| \leq 5} \|\Gamma^\alpha \phi\|_{L^2(\mathbb{R}^3)} \leq C(1+t) \sum_{|\alpha| \leq 4} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^3)},$$

which implies

$$\begin{aligned} & \int_0^T \left(\sum_{1 \leq |\alpha| \leq 5} \|\Gamma^\alpha \phi\|_{L^2(\mathbb{R}^3)} \right) \left(\sum_{|\beta| \leq 2} \|\partial \Gamma^\beta \phi\|_{L^\infty(\mathbb{R}^3)} \right) \left(\sum_{|\gamma| \leq 2} \|\bar{\partial} \Gamma^\gamma \phi\|_{L^\infty(\mathbb{R}^3)} \right) (t) dt \\ & \leq C \epsilon^{\frac{3}{2}} \int_0^T \frac{\sum_{|\alpha| \leq 4} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^3)}}{(1+t)^{\frac{3}{2}}} dt \leq C \epsilon^{\frac{9}{4}}. \end{aligned}$$

Returning to (11.5), we therefore have

$$\sup_{t \in [0, T]} \sum_{|\alpha| \leq 5} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^3)}(t) + \left(\int_0^T \int_{\mathbb{R}^3} \frac{|\bar{\partial} \phi|^2}{(1+|t-|x||)^{1+\delta}} dx dt \right)^{\frac{1}{2}} \leq C(\epsilon + \epsilon^{\frac{3}{2}} + \epsilon^{\frac{9}{4}}).$$

As long as ϵ is sufficiently small, this improves the constant in (11.1) and we are done. \square

As we saw above, the null structure in the nonlinear terms plays a crucial role in establishing global regularity for solutions arising from small data. It is clear from the proof above that it works more generally to semilinear wave equations in $(3+1)$ -dimensions with quadratic nonlinearity satisfying the classical null condition.

12. WEAK NULL CONDITION

We consider the system of equations

$$(12.1) \quad \begin{cases} \square \phi = Q_0(\psi, \psi) \\ \square \psi = (\partial_t \phi)^2. \end{cases}$$

This equation does not obey the classical null condition. More precisely, the nonlinearity $(\partial_t \phi)^2$ is not a null form. In particular, *even if* $\partial_t \phi$ obeys the decay estimates as for the linear wave equation $\partial \psi$ *does not* decay like $\frac{1}{1+t}$ but instead only decays like $\frac{\log(2+t)}{1+t}$ (**Exercise**). Nevertheless, when ψ enters in the nonlinear term, it does so in a null form. The null form provides more decay and therefore global regularity still holds

for small data. Such nonlinear structure is an example of what is more generally known as the **weak null condition** of Lindblad-Rodnianski. We will not discuss systematically the weak null condition, but will just study the example (12.1) given above. We note that a very similar form of this structure will appear again when we discuss the stability of Minkowski space.

Theorem 12.1. *Consider the equation (12.1). Suppose the initial data obey*

$$(\phi_0, \phi_1, \psi_0, \psi_1) \upharpoonright_{\{t=0\}} \in (C_c^\infty(B(0, R)))^4$$

such that

$$\sum_{|\alpha| \leq 5} \|\partial \partial^\alpha (\phi_0, \psi_0)\|_{L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)} + \|\partial^\alpha (\phi_1, \psi_1)\|_{L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)} < \epsilon.$$

Then for every $R > 0$, there exists $\epsilon_0 = \epsilon_0(R) > 0$ sufficiently small such that if $\epsilon \leq \epsilon_0$, the unique solution remains smooth for all time.

Proof. We begin with the bootstrap assumption

$$(12.2) \quad \sup_{t \in [0, T]} \sum_{|\alpha| \leq 5} \|\partial \Gamma^\alpha \psi\|_{L^2(\mathbb{R}^3)}(t) + \sum_{|\alpha| \leq 5} \left(\int_0^T \int_{\mathbb{R}^3} \frac{|\bar{\partial} \Gamma^\alpha \psi|^2}{(1 + |t - |x||)^{1+\delta}} dx dt \right)^{\frac{1}{2}} \leq \epsilon^{\frac{3}{4}} (1 + T)^{\frac{1}{10}}.$$

This implies decay estimates for ψ which are slightly worse than the optimal ones. More precisely, by Theorem 9.4, we have

$$(12.3) \quad \sup_{t \in [0, T]} \sum_{|\alpha| \leq 3} \|(1+t)(1+|t-|x||)^{\frac{1}{2}} \partial \Gamma^\alpha \psi\|_{L^\infty(\mathbb{R}^3)} \leq C \epsilon^{\frac{3}{4}} (1+t)^{\frac{1}{10}}.$$

Using (9.1) together with (12.3), we get

$$(12.4) \quad \sup_{t \in [0, T]} \sum_{|\alpha| \leq 2} (1+t)^{\frac{3}{2}} \|\bar{\partial} \Gamma^\alpha \psi\|_{L^\infty(\mathbb{R}^3)} \leq C \epsilon^{\frac{3}{4}} (1+t)^{\frac{1}{10}}.$$

The key observation is that when we proved the boundedness of energy for the wave map equation, we did not need to use the full strength of the decay estimates. Morally, the presence of the null structure allows us to gain an extra decay rate. We now control

$$\sum_{|\alpha| \leq 5} \int_0^T \|\Gamma^\alpha Q_0(\psi, \psi)\|_{L^2(\mathbb{R}^3)} dt$$

as in the proof of Theorem 11.6. We have two terms, the first is when the factor with fewer Γ 's comes with a good derivative, while the second term is such that the factor with more Γ 's comes with a good derivative. The first can be bounded by

$$\int_0^T \left(\sum_{|\alpha| \leq 5} \|\partial \Gamma^\alpha \psi\|_{L^2(\mathbb{R}^3)} \right) \left(\sum_{|\beta| \leq 2} \|\bar{\partial} \Gamma^\beta \psi\|_{L^\infty(\mathbb{R}^3)} \right) dt \leq C \int_0^T \frac{\epsilon^{\frac{3}{2}} dt}{(1+t)^{\frac{3}{2}-\frac{1}{5}}} \leq C \epsilon^{\frac{3}{2}}.$$

We now control the second term. Let $T_0 = 0$, $T_i = 2^{i-1}$, for $i = 1, \dots, \lfloor \log_2 T \rfloor$, $T_{\lfloor \log_2 T \rfloor + 1} = T$.

$$\begin{aligned} & \int_0^T \left(\sum_{|\alpha| \leq 5, |\beta| \leq 2} \|\bar{\partial} \Gamma^\alpha \psi \partial \Gamma^\beta \psi\|_{L^2(\mathbb{R}^3)} \right) (t) dt \\ & \leq C \sum_{i=0}^{\lfloor \log_2 T \rfloor} \sum_{|\alpha| \leq 5, |\beta| \leq 2} \int_{T_i}^{T_{i+1}} \left(\int_{\mathbb{R}^3} \frac{|\bar{\partial} \Gamma^\alpha \psi|^2}{(1 + |t - |x||)^{1+\delta}} dx \right)^{\frac{1}{2}} \left(\sup_x \left((1 + |t - |x||)^{\frac{1+\delta}{2}} |\partial \Gamma^\beta \psi|(t, x) \right) \right) dt \\ & \leq C \sum_{i=0}^{\lfloor \log_2 T \rfloor} \left(\sum_{|\alpha| \leq 5} \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}^3} \frac{|\bar{\partial} \Gamma^\alpha \psi|^2}{(1 + |t - |x||)^{1+\delta}} dx dt \right)^{\frac{1}{2}} \left(\sum_{|\beta| \leq 2} \left(\int_{T_i}^{T_{i+1}} \left(\sup_x \left((1 + |t - |x||)^{\frac{1+\delta}{2}} |\partial \Gamma^\beta \psi|(t, x) \right) \right)^2 dt \right)^{\frac{1}{2}} \right) \\ & \leq C \epsilon^{\frac{3}{2}} \sum_{i=0}^{\lfloor \log_2 T \rfloor} (1 + T_i)^{\frac{1}{10}} \left(\int_{T_i}^{T_{i+1}} \frac{dt}{(1+t)^{2-\frac{1}{5}-\delta}} \right)^{\frac{1}{2}} \leq C \epsilon^{\frac{3}{2}} \sum_{i=0}^{\lfloor \log_2 T \rfloor} (1 + T_i)^{-\frac{1}{2} + \frac{1}{5} + \frac{\delta}{2}} \leq C \epsilon^{\frac{3}{2}}, \end{aligned}$$

as long as $\delta \in (0, \frac{3}{5})$. Using energy estimates for ϕ , we can then conclude that

$$(12.5) \quad \sup_{t \in [0, T]} \sum_{|\alpha| \leq 5} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^3)}(t) \leq C\epsilon$$

and also

$$(12.6) \quad \sup_{x, t} \sum_{|\alpha| \leq 3} (1+t+|x|)(1+|t-|x||)^{\frac{1}{2}} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^3)}(t) \leq C\epsilon$$

via Klainerman-Sobolev inequality. We now use (12.5) and (12.6) for the energy estimates for ψ to get

$$\begin{aligned} & \sup_{t \in [0, T]} \sum_{|\alpha| \leq 5} \|\partial \Gamma^\alpha \psi\|_{L^2(\mathbb{R}^3)}(t) + \sum_{|\alpha| \leq 5} \left(\int_0^T \int_{\mathbb{R}^3} \frac{|\bar{\partial} \Gamma^\alpha \psi|^2}{(1+|t-|x||)^{1+\delta}} dx dt \right)^{\frac{1}{2}} \\ & \leq C \left(\epsilon + \int_0^T \left(\sum_{|\alpha| \leq 5} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^3)}(t) \right) \left(\sum_{|\beta| \leq 2} \|\partial \Gamma^\beta \phi\|_{L^\infty(\mathbb{R}^3)}(t) \right) dt \right) \\ & \leq C \left(\epsilon + \epsilon^2 \int_0^T \frac{dt}{1+t} \right) \leq C\epsilon \log(2+T). \end{aligned}$$

This improves over (12.2) and we are done. \square

13. ANOTHER DECAY ESTIMATE

By now we know that decay estimates are very important for understanding long time behaviour of solutions to nonlinear equations. Before we proceed further, we prove an additional decay estimate. This estimate is most useful when the energy grows but one still wants to obtain sharp pointwise decay estimate. For a simple application, see Corollary 13.2. This will also play an important role when we discuss the nonlinear stability of Minkowski space.

Proposition 13.1. *Consider the equation*

$$(m^{\alpha\beta} + H^{\alpha\beta}) \partial_{\alpha\beta}^2 \phi = F$$

in $(3+1)$ -dimensions, where m is the Minkowski metric and $H^{\alpha\beta}$ is a symmetric 2-tensor such that

$$\|H\|_{L^\infty(D_t)} < \frac{1}{4}, \quad |H^{uu}| + |H^{uv}| + |H^{uA}| \leq \frac{1+|t-|x||}{10(1+t+|x|)},$$

where $D_t := \{\frac{t}{2} \leq |x| \leq \frac{3t}{2}\}$. Then, we have the following decay estimates:

$$\begin{aligned} & \|(1+t)\partial\phi\|_{L^\infty(\mathbb{R}^3)}(t) \\ & \leq C \sup_{\tau \in [0, t]} \sum_{|\alpha| \leq 1} \|\Gamma^\alpha \phi\|_{L^\infty(\mathbb{R}^3)}(\tau) + C \int_0^t \left((1+\tau)\|F\|_{L^\infty(D_\tau)} + \sum_{|\alpha| \leq 2} (1+\tau)^{-1} \|\Gamma^\alpha \phi\|_{L^\infty(D_\tau)} \right) d\tau. \end{aligned}$$

Proof. Since we allow $\sum_{|\alpha|=1} \|\Gamma^\alpha \phi\|_{L^\infty(D_\tau)}$ on the right hand side, we can control $\bar{\partial}\phi$ on the left hand side by (9.1) in the proof of Corollary 9.8. It therefore suffices to control $\partial_u \phi$. Moreover, it suffices to control $\partial_u \phi$ in the region $\frac{t}{2} \leq |x| \leq \frac{3t}{2}$. We now use the equation to get

$$|4\partial_v \partial_u(r\phi) + rH^{uu} \partial_{uu}^2 \phi| \leq C(r(1+|H|)|\bar{\partial}\phi| + r|H^{uv}||\bar{\partial}\phi| + r|H^{uA}||\bar{\partial}\phi| + r|F|).$$

This implies

$$|(4\partial_v + H^{uu} \partial_u) \partial_u(r\phi)| \leq C \left(\sum_{|\alpha| \leq 2} \frac{|\Gamma^\alpha \phi|}{1+t} + |H^{uu}| |\partial_u \phi| + r|F| \right) \leq C \left(\sum_{|\alpha| \leq 2} \frac{|\Gamma^\alpha \phi|}{1+t} + (1+t)|F| \right).$$

Given a point such that $\frac{t}{2} \leq |x| \leq \frac{3t}{2}$, we integrate the estimate above along the the integral curves of $4\partial_v + H^{uu} \partial_u$ towards the past until it hits the boundary of the set $\{\frac{t}{2} \leq |x| \leq \frac{3t}{2}\}$. This gives the desired bound. \square

One immediate consequence of the estimate above is the following corollary:

Corollary 13.2. *Let F be a smooth function supported in $B(0, t + 1)$ for every $t \geq 0$ such that*

$$\sum_{|\alpha| \leq 5} (1 + T)^{-\frac{1}{10}} \int_0^T \|\Gamma^\alpha F\|_{L^2(\mathbb{R}^3)}(t) dt + \sup_{t,x} (1 + t)^2 \log^2(2 + t) |F(t, x)| \leq C'.$$

Suppose ϕ is a solution to $\square\phi = F$ with compactly supported initial data, then

$$(1 + t) \|\partial\phi\|_{L^\infty(\mathbb{R}^3)}(t) \leq C,$$

for some $C > 0$ depending on initial data and C' .

Proof. Exercise.

□