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Mathematical Tripos, Part III
Functional Analysis

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Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

1 Preliminaries

1.1 Review of linear analysis

1.2 Hilbert spaces and spectral theory

1.3 Some important theorems in Banach spaces

Lemma 1.1 (Risez). *Let X be a normed space. Suppose Y is a closed proper subspace of X, then $\forall\epsilon > 0$, $\exists\ \|x\| = 1$ such that $d(x, Y) = \inf_{y\in Y} \|x - y\| > 1 - \epsilon$.*

Proof. Pick $z \in X \setminus Y$. Since Y is closed, $d(z, Y) > 0$, so there exists $y \in Y$ such that $(1 - \epsilon)\|z - y\| < d(z, Y)$. Let $x = (z - y)/\|z - y\| \in B_X$,

$$d(x, Y) = d\Big(\frac{z}{\|z - y\|}, Y\Big) = \frac{d(z, Y)}{\|z - y\|} > 1 - \epsilon. \tag{1.1}$$

□

Remark 1.1. Let Y be a subspace of X. Suppose there exists a $0 \le \delta < 1$ such that for every $x \in B_X$, there exists a $y \in Y$ with $\|x - y\| \le \delta$. Then $\overline{Y} = X$.

Theorem 1.2. *Let X be a normed space. Then the dimension of X is finite if and only if B_X is compact.*

Proof. ($\Rightarrow$) Since $X \sim l_2^n$, the result follows from Heine-Borel theorem.
($\Leftarrow$) Assume $\dim X = \infty$, we can construct a sequence $(x_n) \in B_X$ such that $\|x_m - x_n\| > 1/2$ for $m \ne n$. This is done by induction: having found $x_1, \cdots, x_n$, we apply Risez's lemma to $Y = span\{x_1, \cdots, x_n\}$, so there exists an $x_{n+1} \in B_X$ such that $d(x_{n+1}, Y) > 1/2$. Note that $x_1 \in B_X$ is arbitrary. Then we are done. □

Theorem 1.3 (Stone-Weierstrass). *Let K be a compact Hausdorff space. Consider $C^{\mathbb{R}}(K) = \{f : \mathbb{R} \to \mathbb{R} : f$ is continuous$\}$ with the sup norm. Suppose A is a subalgebra of $C^{\mathbb{R}}(K)$ that separates the points of K ($\forall x \ne y$ in K, $\exists f \in A$, $f(x) \ne f(y)$) and contains the constant function, then $\overline{A} = C^{\mathbb{R}}(K)$.*

Proof. First we claim that *if we are given two disjoint closed subsets E, F of K, then there exists $f \in A$ such that $-1/2 \le f \le 1/2$ on K, where $-1/4 \le f$ on E and $f \ge 1/4$ on F.* Then we are done. Let $g \in C^{\mathbb{R}}(K)$, with $\|g\|_\infty \le 1$. Then we apply the claim to $E = \{g \le -1/4\}$, $F = \{g \ge 1/4\}$, then $\|f - g\|_\infty \le \frac{3}{4}$. By Risez's lemma, $\overline{A} = C^{\mathbb{R}}(K)$.
Proof of the claim. Fix $x \in E$. For any $y \in F$, $\exists h \in A$ such that $h(x) = 0$, $h(y) > 0$, and $h \ge 0$ on K. There exists an open neighborhood V of y such that $h > 0$ on V. By an easy

compactness argument, we can find a $g = g_x \in A$ such that $g(x) = 0$, $g > 0$ on F, and $0 \le g \le 1$ on K.
There exists $R = R_x > 0$ such that $g > \frac{2}{R}$ on F, and an open neighborhood $U = U_x$ of x such that $g < \frac{1}{2R}$ on U. Do this for every $x \in E$. By compactness, we can find $x_1, \cdots, x_m \in E$ such that $\cup_{i=1}^m U_{x_i} \supset E$. Now we write $g_i = g_{x_i}$, $R_i = R_{x_i}$, $U_i = U_{x_i}$, for $1 \le i \le m$.
For fixed i, on U_i, we have

$$(1 - g_i^n)^{R_i^n} \ge 1 - (g_i R_i)^N = 1 - 2^{-n} \to 1 \tag{1.2}$$

On F, we have

$$(1 - g_i^n)^{R_i^n} \le \frac{1}{(1 + g_i^n)^{R_i^n}} \le \frac{1}{(g_i R_i)^n} \le 2^{-n} \to 0. \tag{1.3}$$

Now we can find an $n_i \in \mathbb{N}$ such that if we let $h_i = 1 - (1 - g_i^{n_i})^{R_i^{n_i}}$, then $h_i \le 1/4$ on U_i and $h_i \ge \left(\frac{3}{4}\right)^{1/m}$ on F.
Now let $h = h_1 \cdot h_2 \cdots h_m$, then $h \in A$ and $0 \le h \le 1$ on K. Note that $h \le 1/4$ on E and $h \ge 3/4$ on F. Finally, let $f = h - 1/2$.

□

Remark 1.2. We have used the **Euler's inequality**:

$$1 - Nx \le (1 - x)^N \le \frac{1}{(1 + x)^N} \le \frac{1}{Nx} \tag{1.4}$$

for $0 < x < 1$ and $N \ge 1$.

Remark 1.3. Stone-Weierstrass fails for complex scalars. In fact, let $\Delta = \{z \in \mathbb{C} : |z| = 1\}$ and $D = int\Delta = \delta^o = \{z \in C : |z| < 1\}$. Consider the **disk algebra**

$$A(\Delta) = \{f \in C(\Delta) : f \text{ is analytic on } D\}. \tag{1.5}$$

Then $A(\Delta)$ is a closed subalgebra of $C(\Delta)$.

Theorem 1.4 (Complex Stone-Weierstrass). *Let K be a compact Hausdorff space. Suppose A is a subalgebra of $C^{\mathbb{C}}(K) = \{f : K \to \mathbb{C} : f \text{ is continuous}\}$ that separates points, contains the constant functions, and are closed under complex conjugation($f \in A \Rightarrow \bar{f} \in A$), then $\overline{A} = C^{\mathbb{C}}(K)$.*

Remark 1.4. There is a more general version for locally compact Hausdorff spaces.

Lemma 1.5 (Open mapping lemma). *Let X be a Banach space, and Y be a normed spaces. Let $T : X \to Y$ be a bounded linear map. Assume that there exists an $M \ge 0$, and $0 \le \delta < 1$ such that $T(MB_X)$ is δ-dense in B_Y. Then T is surjective, that is, for any $y \in Y$ we can find an $x \in X$ such that $y = Tx$, and*

$$\|x\| \le \frac{M}{1 - \delta}\|y\|, \tag{1.6}$$

i.e. $T(\frac{M}{1-\delta}B_X) \supset B_Y$. Moreover, Y is complete.

Definition 1.1. If A and B are subsets of a metric space (M,d), and let $\delta > 0$, then A is δ-*dense* in B if for any $b \in B$, we can find an $a \in A$ such that $d(a,b) \leq \delta$.

Proof. Let $y \in B_Y$, then there exists $x_1 \in MB_X$ such that $\|y - Tx_1\| \leq \delta$. Then there exists an $x_2 \in MB_X$ such that

$$\|\frac{y - Tx_1}{\delta} - Tx_2\| \leq \delta, \tag{1.7}$$

i.e.

$$\|y - Tx_1 - \delta Tx_2\| \leq \delta^2. \tag{1.8}$$

Note that $\frac{y-Tx_1}{\delta} \in B_y$. Continue inductively, we obtain a sequence $\{x_n\}$ in MB_X such that

$$\|y - Tx_1 - T(\delta x_2) - \cdots - T(\delta^{n-1}x_n)\| \leq \delta^n \tag{1.9}$$

for any $n \in \mathbb{N}$. Let $x = \sum_{n=1}^{\infty} \delta^{n-1}x_n$. Since

$$\sum_{n=1}^{\infty} \|\delta^{n-1}x_n\| \leq \delta^{n-1}M = \frac{M}{1-\delta}, \tag{1.10}$$

so the series converges and $\|x\| \leq \frac{M}{1-\delta}$. Now

$$y - Tx = \lim_{n\to\infty} \big(y - \sum_{k=1}^{n} T(\delta^{k-1}x_k)\big) = 0. \tag{1.11}$$

For the last part, let $\tilde{Y}$ be the completion of Y. Consider T as a map $X \to \tilde{Y}$. Since $\overline{B_Y} = B_{\tilde{Y}}$, $T(MB_X)$ is δ-dense in B_Y where $0 \leq \delta < 1$. So T is onto as a map from X to $\tilde{Y}$. Hence $Y = \tilde{Y}$. □

Remark 1.5. If $\overline{T(B_X)} \supset B_Y$, then $T(B_X^o) \supset B_Y^o$.

Quotient spaces Let X be a normed space, $Y \subset X$ a closed subspace. Then X/Y becomes a normed space , where

$$\|x + Y\| = d(x, Y) = \inf_{y\in Y} \|x + y\|. \tag{1.12}$$

(Y closed is needed to show that if $Z \in X/Y$ with $\|z\| = 0$, then $z = 0$)

Proposition 1.6. *Let X be a Banach space and $Y \subset X$ be a closed subspace. Then X/Y is complete.*

Proof. Consider the quotient map $q : X \to X/Y$ defined by $q(x) = x+Y$, then $q \in \mathcal{B}(X, X/Y)$. In fact,

$$\|q(x)\| = \|x + Y\| \leq \|x\| \tag{1.13}$$

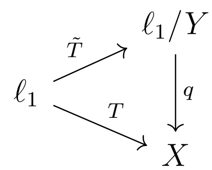
so $\|q\| \leq 1$. Given $x + Y \in B_{X/Y}^o$, there exists $y \in Y$ such that $\|x + y\| < 1$, and $q(x + y) = x + Y$, so $B_{X/Y}^o \subset q(B_X^o)$. Thus $B_{X/Y}^o = q(B_X^o)$ (note that the other direction follows from $\|q\| \leq 1$). In particular, $\overline{q(B_X)} \supset B_{X/Y}$. By open mapping lemma, X/Y is complete. □

Proposition 1.7. *Every separable Banach space is a quotient of ℓ_1, i.e. there exists a closed subspace Y of ℓ_1 such that $\ell_1/Y \simeq X$.*

Proof. Let $\{x_n\}$ be dense in B_X. Define $T : \ell_1 \to X$ by $T(a) = \sum_{n=1}^{\infty} a_n x_n$, where $a = \{a_n\}$. Note that

$$\sum_{n=1}^{\infty} \|a_n x_n\| \le \sum_{n=1}^{\infty} |a_n| = \|a\|_1 < \infty, \tag{1.14}$$

so $T \in \mathcal{B}(\ell_1, X)$ with $\|T\| \le 1$. Thus $T(B^o_{\ell_1}) \subset B^o_X$. Since $\{x_n : n \in \mathbb{N}\} \subset T(B_{\ell_1})$, $B_X \subset \overline{T(B_{\ell_1})}$. By the open mapping lemma, $B^o_X \subset T(B^o_{\ell_1})$. Thus $B^o_X = T(B^o_{\ell_1})$.
Now let $Y = \ker T$ which is a closed subspace of ℓ_1. Let $\tilde{T}$ be the unique linear map such that



commutes, where $q : \ell_1 \to \ell_1/Y$ is the quotient map. Note that $\tilde{T}$ is a bijection

$$\tilde{T}(B^o_{\ell_1/Y}) = \tilde{T}\big(q(B^o_{\ell_1})\big) = T(B^0_{\ell_1}) = B^o_X. \tag{1.15}$$

Hence $\tilde{T}$ is an isometric isomorphism. □

Recall that a topological space K is **normal** if whenever E and F are disjoint closed subsets of K, there exist disjoint open sets U and V such that $E \subset U$ and $F \subset V$. For example, a compact Hausdorff space is normal.

Lemma 1.8 (Uryson's)**.** *Let K be a normal space, and let E and F be disjoint closed subsets of X, then there exists a continuous function $f : K \to [0,1]$ such that $f = 0$ on E and $f = 1$ on F.*

So $C(K)$ separates the points of K for a compact Hausdorff space K.

Theorem 1.9 (Tietze extension theorem)**.** *Let K be a normal topological space, and let L be a closed subspace. Suppose $g : L \to \mathbb{R}$ is continuous and bounded, then there exists a continuous and bounded function $f : K \to \mathbb{R}$ such that $f\big|_L = g$ and $\|f\|_\infty = \|g\|_\infty$.*

Remark 1.6. Assume $f : K \to \mathbb{R}$ is continuous, $f|_L = g$. Define

$$\phi(\lambda) = \begin{cases} \lambda & \text{if } |\lambda| \le \|g\|_\infty, \\ \frac{\lambda}{|\lambda}\|g\|_\infty & \text{if } |\lambda| > \|g\|_\infty. \end{cases} \tag{1.16}$$

Then ϕ is continuous with $(\phi \circ f)|_L = g$ and $\|\phi \circ f\|_\infty = \|g\|_\infty$.

Proof. Let $X = C_b(K) = \{f : K \to \Re : f \text{ is continuous and bounded}\}$, then X is a Banach space with sup norm. Let $Y = C_b(L)$ and consider the map $R : X \to Y$ defined by $R(f) = f|_L$. Clearly $R \in \mathcal{B}(X, Y)$ and $\|R\| \le 1$. We have to show that R is onto. In fact, we will show $R(B_X) = B_Y$.

Let $g \in B_Y$, apply the Uryson's lemma with $E = \{g \leq -1/3\}$ and $F = \{g \geq 1/3\}$ to obtain a continuous function $f : K \to [-1/3, 1/3]$ such that $f = -1/3$ on E and $f = 1/3$ on F. Then $f \in \frac{1}{3}B_X$ and

$$\|Rf - g\|_\infty \leq 2/3. \tag{1.17}$$

So $R(\frac{1}{3}B_X)$ is $\frac{2}{3}$-dense in B_Y. By the open mapping lemma,

$$R\Big(\frac{\frac{1}{3}B_X}{1-2/3}\Big) \supset B_Y, \tag{1.18}$$

i.e. $R(B_X) \supset B_Y$. □

Remark 1.7. The complex version is also true.

1.4 Review of measure theory

2 Hahn-Banach theorems and LCS

2.1 The Hahn-Banach theorems

For a normed space X, we write X^* for its *dual space*, i.e.

$$X^* = \mathcal{B}(X, \mathbb{R}) \tag{2.1}$$

(or $\mathbb{C}$ instead of $\mathbb{R}$), which is the space of all bounded linear functionals on X. X^* is always complete with the operator norm

$$\|f\| = \sup\{|f(x)| : x \in B_X\}. \tag{2.2}$$

So $|f(x)| \leq \|f\| \cdot \|x\|$ for all $x \in X$ and $f \in X^*$. We will use $< x, f >$ as notation for $f(x)$.

Definition 2.1. Let X be a real vector space. A functional p is called

- **positive homogeneous** if $p(tx) = tp(x)$, for all $t \geq 0$ and $x \in X$.
- **subadditive** if $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

Theorem 2.1 (Hahn-Banach)**.** *Let X be a real vector space, and p be a positive homogeneous, subadditive functional on X. Let Y be a subspace of X and $g : Y \to \mathbb{R}$ be a linear functional such that $g(y) \leq p(y)$ for all $y \in Y$. Then there exists a linear functional $f : X \to \mathbb{R}$ such that $f|_Y = g$ and $f(x) \leq p(x)$ for all $x \in X$.*

Recall: let $(P, \leq)$ be a non-empty poset. A *chain* is a subset $A \subset P$ which is linearly ordered by $\leq$. An element $x \in P$ is an *upper bound* for a subset A if $a \leq x$ for all $a \in A$. An element $x \in P$ is a *maximal element* of P if whenever $x \leq y$ for some $y \in P$, then $y = x$. We will use the Zorn's lemma in the proof of the Hahn-Banach theorem.

Lemma 2.2 (Zorn's)**.** *Let $P \neq \emptyset$. If every non-empty chain in P has an upper bound, then P has a maximal element.*

Proof of Theorem 2.1. Let

$$P = \{(Z,h):\ h : Z \to \mathbb{R} \text{ is a linear such that } h|_Y = g,\ h(z) \le p(z) \forall z \in Z,\ \}, \tag{2.3}$$

where Z is a subspace of X, Y is a subspace of Z. Then P is non-empty since $(Y, g) \in P$. Let $\{(Z_i, h_i) : i \in I\}$ be a non-empty chain. Let $Z = \bigcup_{i \in I} Z_i$, and define $h : Z \to \mathbb{R}$ by

$$h(z) = h_i(z) \quad \text{for } z \in Z_i,\ i \in I. \tag{2.4}$$

Then (Z, h) is an upper bound for the chain. (Note that $(Z_1, h_1) \le (Z_2, h_2)$ iff $Z_1 \subset Z_2$ and $h_1 = h_2|_{Z_1}$)
By Zorn's lemma, there exists a maximal element (W, f). We need show $W = X$. Suppose not, pick an $x_0 \in X - W$. Let $W_1 = W \oplus \mathbb{R}x_0$ and define $f_1 : W_1 \to \mathbb{R}$ by

$$f_1(x + \lambda x_0) = f(x) + \lambda\alpha \tag{2.5}$$

where $x \in W$, $\lambda \in \mathbb{R}$, and α is to be determined. We want

$$f_1(x + \lambda x_0) \le p(x_\lambda x_0) \tag{2.6}$$

for all $x \in W$ and $\lambda \in \mathbb{R}$. By positive homogeneity, it suffices to have

$$f_1(x + x_0) \le p(x + x_0) \quad \text{and} \quad f_1(x - x_0) \le p(x - x_0), \tag{2.7}$$

which is

$$f(x) + \alpha \le p(x + x_0) \quad \text{and} \quad f(x) - \alpha \le p(x - x_0). \tag{2.8}$$

By rearranging the term, this is equivalent to

$$f(y) - p(y - x_0) \le \alpha \le p(x + x_0) - f(x) \tag{2.9}$$

for any $x, y \in W$. Hence α exists iff

$$f(y) - p(y - x_0) \le p(x + x_0) - f(x) \tag{2.10}$$

for any $x, y \in W$. But this always holds since

$$f(x) + f(y) = f(x + y) \le p(x + y) \le p(x + x_0) + p(y - x_0). \tag{2.11}$$

Therefore, $(W_1, f_1) \in P$ which is strictly bigger that (W, f). But this contradict the maximality of (W, f). Hence $W = X$. □

Definition 2.2. A **seminorm** on a real or complex vector space X is a function $p : X \to \mathbb{R}$ (or $\mathbb{C}$) such that

- $p(x) \ge 0$ for all $x \in X$,
- $p(\lambda x) = |\lambda| p(x)$ for all $x \in X$ and λ is a scalar,
- $p(x + y) \le p(x) + p(y)$ for all $x, y \in X$.

Remark 2.1. We have the following inclusions:

$$\text{positive homogeneous+subadditive } \to \text{seminorm} \to \text{norm} \tag{2.12}$$

Theorem 2.3 (Hahn-Banach)**.** *Let X be a real or complex vector space, and p a seminorm on X. Given a subspace $Y \subset X$, and a linear functional g on Y such that*

$$|g(y)| \leq p(y) \quad \forall y \in Y. \tag{2.13}$$

Then g extends to a linear functional f on X such that

$$|f(x)| \leq p(x) \quad \forall x \in X. \tag{2.14}$$

Proof. (*Real scalar*) We have $g(y) \leq p(y)$ $\forall y \in Y$. By Theorem 2.1, there exists $f : X \to \mathbb{R}$ such that $f|_Y = g$ and $f(x) \leq p(x)$ for all $x \in X$. Since

$$-f(x) = f(-x) \leq p(-x) = p(x) \quad \forall x \in X, \tag{2.15}$$

we have $|f(x)| \leq p(x)$.
(*Complex scalar*) Consider $g_1 = Re(g)$, that is, $g_1(y) = Re\big(g(y)\big)$, which is a real linear map $Y \to \mathbb{R}$ and

$$|g_1(y)| \leq |g(y)| \leq p(y) \quad \forall y \in Y. \tag{2.16}$$

By the real case, there exists a real linear functional $f_1 : X \to \mathbb{R}$ such that $f_1|_Y = g_1$ and $|f_1(x)| \leq p(x)$ for all $x \in X$. Now we seek a complex linear functional $f : X \to \mathbb{C}$ such that $Re(f) = f_1$. In fact, such an f is unique. Write $f(x) = f_1(x) + if_2(x)$. Note that

$$f(x) = -if(ix) = -if_1(ix) + f_2(ix), \tag{2.17}$$

so

$$f(x) = f_1(x) - if_1(ix). \tag{2.18}$$

Define f by this formula, so f is real-linear and $f(ix) = if(x)$ for all $x \in X$. Hence $f : X \to \mathbb{C}$ is complex-linear and $Re(f) = f_1$. Note that now

$$Re(f|_Y) = f_1|_Y = g_1 = Re(g). \tag{2.19}$$

By uniqueness, $f|y = g$. Given $x \in X$, choose a $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that

$$|f(x)| = \lambda f(x) = f(\lambda x) = f_1(\lambda x) \leq p(\lambda x) = p(x). \tag{2.20}$$

□

Remark 2.2. If X is a complex normed space, then $(X^*)_{\mathbb{R}} \to (X_{\mathbb{R}})^*$, $f \mapsto Re(f)$ is an isometric isomorphism (real linear).

Corollary 2.4. *Let $x_0 \in X$, then there exists a linear functional on X such that $f(x_0) = p(x_0)$, and $|f(x)| \leq p(x)$ $\forall x \in X$.*

Proof. Let $Y = span\{x_0\}$, and define $g(\lambda x_0) = \lambda p(x_0)$ for all λ. Then g is a linear functional on Y. By Theorem 2.3, g extends to a linear functional f on X such that $|f(x)| \leq p(x)$ $\forall x \in X$, and $f(x_0) = g(x_0) = p(x_0)$. □

Theorem 2.5 (Hahn-Banach)**.** *Let X be a normed space, then*

- *If Y is a subspace of X, $g \in Y^*$, then there exists $f \in X^*$ such that $f|_Y = g$ and $\|f\| = \|g\|$.*
- *If $x_0 \in X$ and $x_0 \neq 0$, then there exists an $f \in S_{X^*}$ such that $f(x_0) = \|x_0\|$.*

Proof. a) Define $p(x) = \|g\| \cdot \|x\|$. Then this is a seminorm on X. Since $\|g(y)\| \leq \|g\| \cdot \|y\| = p(y)$, by Theorem 2.3 there exists linear functional f on X such that $f|_Y = g$ and $|f(x)| \leq p(x) = \|g\| \cdot \|x\|$ for all $x \in X$. Hence $f \in X^*$ with $\|f\| \leq \|g\|$. So $\|f\| = g\|$.

b) Let $Y = span\{x_0\}$, and define $g : Y \to scalar$ by $g(\lambda x_0) = \lambda \|x_0\|$. Then $g \in Y^*$ and $\|g\| = 1$. By a), there exists an $f \in X^*$ with $f|_Y = g$ and $\|f\| = \|g\| = 1$. In particular, $f(x_0) = g(x_0) = \|x_0\|$. □

Remark 2.3. a) can be viewed as a linear version of Tietze's extension theorem.

Remark 2.4. b) says that X^* separates the points of X: if $x \neq y$ in X, apply b) to $x_0 = x - y$. Thus there are plenty of linear functionals on X.

Remark 2.5. The functional f in b) is call the *norming functional for x_0* or *the supporting functional at x_0*. It shows that

$$\|x_0\| = \sup\{|f(x_0)| : f \in B_{X^*}\}. \tag{2.21}$$

In complex plane, we can replace $f(x_0)$ by $Re\big(f(x_0)\big)$. Assume that $\|x_0\| = 1$, then the half-space $\{x \in X : f(x) \leq 1\}$ (or $\{x \in X : Re(f(x)) \leq 1\}$ in the complex case) is a sort of tangent to B_X at x_0.

2.2 Bidual

For a normed space X, we write X^{**} for $(X^*)^* = \mathcal{B}(X, scalar)$, which is the Banach space of all bounded linear functionals on X^* with the operator norm. For $x \in X$, we define $\hat{x} : X^* \to \mathbb{R}(\text{or } \mathbb{C})$ by $\hat{x}(f) = f(x)$ (*evaluation* at x). Then $\hat{x} \in X^{**}$ and $\|\hat{x}\| \leq \|x\|$. The map $x \mapsto \hat{x} : X \to X^{**}$ is called the *canonical embedding.*

Theorem 2.6. *The canonical embedding defined above is an isometric isomorphism of X into X^{**}.*

Proof. For $x \in X$, it's easy to show that $\hat{x}$ is linear. Since

$$|\hat{x}(f)| \leq |f(x)| \leq \|f\| \cdot \|x\| \quad \forall f \in X^*, \tag{2.22}$$

so $x \in X^{**}$ and $\|\hat{x}\| \leq \|x\|$. By Theorem 2.5, there exists $f \in B_{X^*}$ such that $\|x\| = f(x)$. So

$$\|\hat{x}\| \geq |\hat{x}(f)| = \|x\| \tag{2.23}$$

Therefore, $\|\hat{x}\| = \|x\|$. Clearly, the map $x \mapsto \hat{x}$ is linear. □

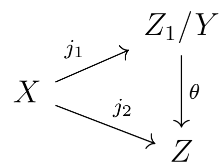
Remark 2.6. Using the bracket notation, we have

$$< f, \hat{x} >=< x, f >= f(x) \tag{2.24}$$

for $x \in X$, $f \in X^*$.

Remark 2.7. The image $\hat{X} = \{\hat{x} : x \in X\}$ of the canonical embedding in X^{**} is closed iff X is complete.

Remark 2.8. In general, the closure of $\hat{X}$ in X^{**} is a Banach space of which X is a dense subspace. So we proved that any normed space X has a completion which is a pair (Z, j) where Z is a Banach space , and $j : X \to Z$ is isometric such that $\overline{j(X)} = Z$. The completion is unique up to isomorphisms. If (Z_1, j_1) and (Z_2, j_2) are both completions, then there exists a unique isometric isomorphism $\theta : Z_1 \to Z_2$ such that the following diagram



commutes, i.e. $\theta \circ j_1 = j_2$.

Definition 2.3. A normed space X is **reflexive** if the canonical embedding of X into X^{**} is surjective, i.e. $\hat{X} = X^{**}$.

By definition a reflexive space must be complete.

Example 2.1. The spaces ℓ_p for $1 < p < \infty$, Hilbert spaces, and finite-dimensional spaces are all reflexive.

Example 2.2. The spaces c_0, ℓ_1, $L_1[0, 1]$ are not reflexive.

Remark 2.9. There are Banach spaces X with $X \simeq X^{**}$ which are not reflexive. So for $1 < p < \infty$, it is not sufficient to say that $\ell_p^{**} \simeq \ell_q^* \simeq \ell_p$ (where $\frac{1}{p} + \frac{1}{q} = 1$) implies ℓ_p is reflexive. One also has to verify that this isomorphism is indeed the canonical embedding.

2.3 Dual operators

Recall that for normed linear spaces X, Y, we denote the space of bounded linear maps $T : X \to Y$ by $\mathcal{B}(X, Y)$. This is a normed space in the operator norm:

$$\|T\| = \sup\{\|Tx\| : x \in B_X\}. \tag{2.25}$$

Moreover, $\mathcal{B}(X, Y)$ is complete if and only is Y is.
We define the **dual operator** of T, $T^* : Y^* \to X^*$ by $T^*(g) = g \circ T$ for $g \in Y^*$, i.e. $T * (g)(x) = g(Tx)$ for $x \in X$, $g \in Y^*$. In bracket notation,

$$< x, T^* g >=< Tx, g > . \tag{2.26}$$

T^* is well-defined since the composite of continuous linear maps is continuous and linear. Moreover, $T* \in \mathcal{B}(Y^*, X^*)$ and

$$\|T^*\| = \sup_{g \in B_{Y^*}} \|T^*g\| \tag{2.27}$$
$$= \sup_{g \in B_{Y^*}} \sup_{x \in B_X} \|g \circ T(x)\| \tag{2.28}$$
$$= \sup_{x \in B_X} \sup_{g \in B_{Y^*}} \|g(Tx)\| \tag{2.29}$$
$$= \sup_{x \in B_X} \|(Tx)\| = \|T\|. \tag{2.30}$$

Example 2.3. Let $1 < p < \infty$, define $T : \ell_p \to \ell_p$ to be the *right shift operator* by $T(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$, then $T^* : \ell_p^* \simeq \ell_q \to \ell_q \simeq \ell_p^*$ is the *left shift operator.*

Properties of dual operators

- $(Id_X)^* = Id_{X^*}$.
- When X and Y are Hilbert spaces, the dual operator T^* corresponds the adjoint of T by identifying X^* and Y^* with X and Y respectively.
- $(\lambda S + \mu T)^* = \lambda S^* + \mu T^*$ for scalars λ, μ, and $S, T \in \mathcal{B}(X, Y)$. In fact,

$$< x, (\lambda S + \mu T)^*(g) > =< (\lambda S + \mu T)x, g > \tag{2.31}$$
$$= \lambda < Sx, g > +\mu < Tx, g > \tag{2.32}$$
$$= \lambda < x, S^*g > +\mu < x, T^*g > \tag{2.33}$$
$$=< x, (\lambda S^* + \mu T^*)g > . \tag{2.34}$$

  Note that there is no complex conjugation here which is different from adjoints in Hilbert spaces. This is due to the fact that the identification of a Hilbert space with its dual is conjugate linear in the complex case.

- $(ST)* = T^*S^*$, where $T \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$.
- If $X \sim Y$, then $X^* \sim Y^*$.

Remark 2.10. The map $T \mapsto T^*$ is an isometric isomorphism of $\mathcal{B}(X, Y)$ *into* $\mathcal{B}(Y^*, X^*)$.

Remark 2.11. We have $\widehat{Tx} = T^{**}\hat{x}$. In fact, let $x \in X$, $g \in Y^*$,

$$< g, \widehat{Tx} > =< Tx, g > \tag{2.35}$$
$$=< x, T^*g > \tag{2.36}$$
$$=< T^*g, \hat{x} > \tag{2.37}$$
$$=< g, T^{**}\hat{x} > . \tag{2.38}$$

Hence the following diagram commutes,

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow{\pi} & & \downarrow{\pi} \\ X^{**} & \xrightarrow{T^{**}} & Y^{**} \end{array}$$

where π denotes the canonical embedding.

Theorem 2.7. *If X^* is separable, then so is X.*

Proof. Let $\{x_n^* : n \in \mathbb{N}\}$ be a dense subset of S_{X^*}, then for each n, we can pick an $x_n \in B_X$ such that $x_n^*(x_n) > \frac{1}{2}$. Let $Y = \overline{span}\{x_n\}_{n\in\mathbb{N}}$, we claim that $Y = X$. Suppose not, take $x_0 \in X \setminus Y$. Since Y is closed, $d(x_0, Y) > 0$. Let $Z = span(Y \cup \{x_0\})$. Define $g : Z \to scalar$ by

$$g(y + \lambda x_0) = \lambda d(x_0, Y) \tag{2.39}$$

for scalar λ and $y \in Y$. Observe that

$$|g(y + \lambda x_0)| = |\lambda| d(x_0, Y) \leq |\lambda| \cdot \|\frac{y}{\lambda} + x_0\| = \|y + \lambda x_0\| \tag{2.40}$$

for $y \in Y$ and $\lambda \neq 0$. Hence $g \in Z^*$ with $\|g\| \leq 1$. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in Y such that $\lim_{n\to\infty} \|y_n + x_0\| = d(x_0, Y)$, then it follows that

$$\lim_{n\to\infty} \frac{g(y_n + x_0)}{\|y_n + x_0\|} = 1, \tag{2.41}$$

and therefore $\|g\| = 1$. By Theorem 2.5, there exists $f \in X^*$ such that $f\big|_Z = g$ and $\|f\| = 1$. Now we can find an n such that $\|f - x_n^*\| < \frac{1}{100}$, but then

$$\frac{1}{2} < |x_n^*(x_n)| = |\big(x_n^* - f\big)(x_n)| < \frac{1}{100} \tag{2.42}$$

which yields a contradiction. □

Remark 2.12. The converse is false. For example, ℓ_1 is separable but ℓ_∞ is not.

Theorem 2.8. *Every separable Banach space X is isometrically isomorphic to a subspace of ℓ_∞, i.e. $X \hookrightarrow \ell_\infty$.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a dense subset in X. For each n, pick an x_n^* in S_{X^*} such that $x_n^*(x_n) = \|x_n\|$ (WLOS assume $X \neq \{0\}$). Define $T : X \to \ell_\infty$ by

$$T(x) = \Big(x_n^*(x)\Big)_{n=1}^{\infty}. \tag{2.43}$$

Since

$$|x_n^*(x)| \leq \|x_n^*\| \cdot \|x\| \leq \|x\|, \quad \forall n \in \mathbb{N}, \tag{2.44}$$

$Tx \in \ell_\infty$ and $\|Tx\|_\infty \leq \|x\|$. Clearly, T is linear. Given $x \in X$. We can find an sequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ such that $x_{n_k} \to x$. Observe that

$$\|x_{n_k}^*(x)\| \geq \|x_{n_k}\| - \|x_{n_k}^*(x - x_{n_k})\| \geq \|x_n\| - 2\|x - x_n\|. \tag{2.45}$$

Letting $n_k \to \infty$, we can find an n_j such that $\|x_{n_j}\| \geq \|x\| - \epsilon$ for any given $\epsilon > 0$. Taking supremum over n, we get $\|Tx\|_\infty \geq \|x\|$. □

Remark 2.13. Let $\mathcal{S}$ be the class of all separable Banach spaces, then ℓ_∞ is *isometrically universal* for $\mathcal{S}$. Note that $\ell_\infty \notin \mathcal{S}$. Question: does there exist a universal $Z \in \mathcal{S}$ for $\mathcal{S}$. The answer is yes and we will see it later.

Remark 2.14. Let $\mathcal{SR}$ be the class of separable reflexive spaces. Question: does there exist a universal $Z \in \mathcal{SR}$ for $\mathcal{SR}$. The answer is no, and it turns out to be much harder.

Theorem 2.9 (Vector-valued Liouville theorem)**.** *Let X be a complex Banach space, and $f : \mathbb{C} \to X$ is an analytic and bounded function. Then f is constant.*

Proof. Note: f is *analytic* means that $\lim_{z\to w} \frac{f(z)-f(w)}{z-w}$ exists for all $w \in \mathbb{C}$. f is *bounded* means that there exists $M > 0$ such that $\|f(z)\| \leq M$ for all $z \in \mathbb{C}$.
Now we return to the proof of the theorem. Let $\phi \in X^*$, and consider the function $\phi \circ f : \mathbb{C} \to \mathbb{C}$. Since ϕ is continuous and linear, the limit of

$$\frac{\phi\big(f(z)\big) - \phi\big(f(w)\big)}{z-w} \tag{2.46}$$

exists and equals to $\phi\big(f'(w)\big)$. Hence $\phi \circ f$ is analytic on $\mathbb{C}$. Also,

$$|\phi \circ f(z)| \leq \|\phi\| \cdot \|f(z)\| \leq M\|\phi\| \tag{2.47}$$

for all $z \in \mathbb{C}$. By the scalar Liouville's theorem, $\phi \circ f$ is constant, so $\phi\big(f(z) - f(0)\big) = 0$ for all $z \in \mathbb{C}$ and $\phi \in X^*$. By Theorem 2.5, X^* separates the points of X, and therefore $f(z) - f(0) = 0$ for all $z \in \mathbb{C}$. □

2.4 Locally convex spaces

A **locally convex space**(LCS) is a real or complex vector space with a family $\mathcal{P}$ of seminorms on X (be a pair $(X, \mathcal{P})$) that separates the points of X in the sense that for every $x \in X$ with $x \neq 0$, there is a seminorm $p \in \mathcal{P}$ with $p(x) \neq 0$.
The family $\mathcal{P}$ defines a *topology* on X: a set $U \subset X$ is *open* if and only if for all $x \in U$, there exist $n \in \mathbb{N}$, $p_1, \cdots, p_n \in \mathcal{P}$, and $\epsilon > 0$ such that

$$\{y \in X : p_k(y-x) < \epsilon \ (k = 1, \cdots, n)\} \subset U. \tag{2.48}$$

An alternative definition is $\bigcap\{p^{-1}(0) : p \in \mathcal{P}\} = \{0\}$.

Remark 2.15. Addition and scalar multiplication is continuous.

Remark 2.16. The topology of X is Hausdorff as $\mathcal{P}$ separates the points of X.

Remark 2.17. If $Y \subset X$ is a subspace, then $\mathcal{P}_Y = \{p|_Y : p \in \mathcal{P}\}$ is a family of seminorms on Y. The topology of LCS $(Y, \mathcal{P}_Y)$ is the subspace topology on Y induced by X.

Remark 2.18. A sequence $x_n \to x$ in X if and only if $p(x_n) \to p(x)$ for all $p \in \mathcal{P}$. (The same holds for nets.)

Remark 2.19. Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of seminorms on X, both of which separate the points of X. We say $\mathcal{P}$ and $\mathcal{Q}$ are *equivalent* if they induce the same topology, and we write $\mathcal{P} \sim \mathcal{Q}$ in this case.

The topology of a locally convex space $(X, \mathcal{P})$ is *metrizable* if and only if there exist countable $\mathcal{Q}$ with $\mathcal{Q} \sim \mathcal{P}$.

Definition 2.4. A *Fréchet space* is a complete metrizable locally convex space. In particular, all Banach spaces are Fréchet spaces.

Example 2.4. Every normed space $(X, \|\cdot\|)$ is a LCS with $\mathcal{P} = \{\|\cdot\|\}$.

Example 2.5. Let U be a non-empty, open subset of $\mathbb{C}$, and let $\mathcal{O}(U)$ denote the space of analytic functions $f : U \to C$. For a compact subset $K \subset U$ and $f \in \mathcal{O}(U)$, set $p_K(f) = \sup_{z\in K} |f(z)|$ and $\mathcal{P} = \{p_K : K \subset U, \text{ and } K \text{ compact }\}$. Then $(\mathcal{O}, \mathcal{P})$ is a locally convex space whose topology is the topology of local uniform convergence.
There exists compact sets $K_n \subset U, n \in \mathbb{N}$, such that $K_n \subset int(K_{n+1})$ and $U = \bigcup_n K_n$. Then $\{p_{K_n} : n \in \mathbb{N}\}$ is countable and equivalent to $\mathcal{P}$. Hence $(\mathcal{O}, \mathcal{P})$ is metrizable and in fact it is a Fréchet space.
The topology of local uniform convergence is not *normable* because it cannot be induced by a norm. This follows, for example, from *Montel's theorem* : given a sequence $\{f_n\}$ in $\mathcal{O}(U)$ such that $\{f_K\}$ is bounded in $\big(C(K), \|\cdot\|\big)$ for every compact $K \subset U$, there is a subsequence converges locally uniformly.

Theorem 2.10 (Montel's theorem). *If $\{f_n\} \subset \mathcal{O}(U)$ is uniformly bounded on compact sets, then there exists a subsequence of $\{f_n\}$ converges locally uniformly.*

3 Risez Representation theorem

Letting K be a compact and Hausdorff space, then

$$C(K) = \{f : K \to \mathbb{C} : f \text{ is continuous}\} \tag{3.1}$$

is a complex Banach space with the sup norm

$$\|f\| = \|f\|_\infty = \sup\{|f(x)| : x \in K\}. \tag{3.2}$$

Define

$$C^{\mathbb{R}}(k) = \{f : K \to \mathbb{R} : f \text{ is continuous}\} \tag{3.3}$$

which is a real Banach space. Similarly, we define

$$C^+(K)* = \{f \in C^{\mathbb{R}}(K) : f(x) \geq 0, \forall x \in K\}. \tag{3.4}$$

Next, we consider the dual spaces related to the previous spaces. Define $M(K)$ to be the dual of $C(K)$

$$M(K) = C(K)^* = \mathcal{B}(C(K), \mathbb{C}) \tag{3.5}$$

If $\phi \in M(K)$, we have the usual operator norm

$$\|\phi\| = \sup\{|\phi(f)| : f \in C(K), \|f\| \leq 1\}. \tag{3.6}$$

Similarly, we define

$$M^{\mathbb{R}}(K) = \{\phi \in M(K) : \phi(f) \in \mathbb{R}, \forall f \in C^{\mathbb{R}}(K)\} \tag{3.7}$$
$$M^{+}(K) = \{\phi : C(K) \to \mathbb{C} : \phi \text{ is linear, and } \phi(f) \geq 0, \forall f \in C^{+}(K)\}. \tag{3.8}$$

The elements of $M^+(K)$ are called **positive linear functionals**.

3.1 Risez representation theorem

Theorem 3.1 (Risez representation). *For every $\phi \in M^+(K)$, there exists a unique finite Borel measure μ such that*

$$\phi(f) = \int_K f \, d\mu, \qquad \forall f \in C(K). \tag{3.9}$$

3.2 L^p spaces

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Fix $1 \leq p < \infty$. $L^p(\Omega, \mathscr{F}, \mu)$ or L^p is the real or complex vector space of measurable functions $f : \Omega \to \mathbb{R}$(or $\mathbb{C}$) such that

$$\int_\Omega |f|^p \, d\mu \leq \infty. \tag{3.10}$$

L^p is a normed space with the norm

$$\|f\|_p = \Big(\int_\Omega |f|^p \, d\mu \Big)^{\frac{1}{p}}, \tag{3.11}$$

provided we identify f, g if $f = g$ a.e. on Ω, i.e. $N = \{x \in \Omega : f(x) = g(x)\}$ is a null set ($\mu(N) = 0$).t $L^p(\Omega, \mathscr{F}, \mu)$ is complete, where $1 \leq p \leq \infty$.
$\| \cdot \|_p$ is a seminorm on L^p. If $\| \cdot \|$ is a seminorm on a vector space X, then $N = \{x \in X : \|x\| = 0\}$ is a subspace. Then $\|x + N\| = \|x\|$ defines a norm on X/N.

The case $p = \infty$ L^∞ is the space of *essentially bounded* measurable scalar-valued functions f on Ω, i.e. there exists a null set $N \subset \sigma$ such that f is bounded on $\Omega \setminus N$, and we define

$$\|f\|_\infty = \operatorname{ess\,sup} |f| = \inf\{\sup_{\Omega \setminus N} |f| : N \subset \Omega, N \subset \Omega \text{ is a null set}\}. \tag{3.12}$$

With this norm, L^∞ becomes a normed space.

Theorem 3.2. *$L^p(\Omega, \mathcal{F}, \mu)$, $1 \leq p \leq \infty$ is complete.*

Proof. 1) $1 \leq p < \infty$.

Let $\{f_n\}$ be a sequence in L^p such that $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$. We will show that $\sum_{i=1}^{\infty} f_k$ converges in L^p. Define $S_n = \sum_{k=1}^{n} |f_k|$. Let $S = \sum_{k=1}^{\infty} |f_k|$, notice that this may take the value ∞. Suppose that $S = \infty$ on some $A \subset \mathcal{F}$ with $\mu(A) > 0$. Fix $L > 0$, then $S_n^p \wedge L \nearrow L$ on A. By the monotone convergence theorem,

$$\int_A (S_n^p \wedge L) d\mu \to \int_A L = \mu(A)L. \tag{3.13}$$

Since

$$\|S_n\|_p \leq \sum_{k=1}^{n} \|f_k\|_p = \sum_{k=1}^{\infty} \|f_k\|_p \overset{def}{=} M, \tag{3.14}$$

We have

$$\int_A (S_n^p \wedge L) d\mu \leq \|S_n\|_p^p \leq M^p \tag{3.15}$$

for all n, which implies that $\mu(A)L \leq M^p$ for all L. We have a contradiction.
Hence $S < \infty$ a.e. (WLOS, suppose $S < \infty$ everywhere). Now $S_n^p \nearrow S^p$, by the monotone convergence theorem

$$\int S^p = \lim \int S_n^p \leq M^p, \tag{3.16}$$

so $S^p \in L^1$.
Since $S < \infty$ on Ω, we can define $f = \sum_{k=1}^{\infty}$. Since $|\sum_{k=1}^{n} f_k - f|^p \to 0$ and $|\sum_{k=1}^{n} f_k - f|^p \leq 2S^p \in L^1$, by the dominate convergence theorem,

$$\int_\Omega |\sum_{k=1}^{n} f_k - f|^p d\mu \to 0 \tag{3.17}$$

as $n \to \infty$. So $f \in L^p$ and $\sum_{k=1}^{n} \to f$ in L^p. □

4 Weak Topologies

4.1 General weak topologies

Let X be a set, $\mathscr{F}$ be a family of functions such that for all $f \in \mathscr{F}$, f is a function from X to Y_f, where each Y_f is a topological space.

Definition 4.1. The **weak topology on** X generated by $\mathscr{F}$, denoted by $\sigma(X, \mathscr{F})$, is the smallest topology on X which makes each $f \in \mathscr{F}$ be continues.

Remark 4.1. A **sub-base** for $\sigma(X, \mathscr{F})$ is

$$S = \{f^{-1}(U) : f \in \mathscr{F}, \text{and } U \text{ is open in } Y_f\}, \tag{4.1}$$

that is, $\sigma(X, \mathscr{F})$ consists of arbitrary unions of finite intersections of elements of S. More generally, if S_f is a sub-base for the topology of Y_f, then $\{f^{-1}(U) : f \in \mathscr{F}, U \in S_f\}$ is also a sub-base for $\sigma(X, \mathscr{F})$.

Remark 4.2. $V \subset X$ is open ($V \in \sigma(X, \mathscr{F})$) means that for every $x \in V$, there exist $n \in \mathbb{N}$, $f_1, f_2, \cdots, f_n \in \mathscr{F}$, and open sets U_i in Y_{f_i} for $i = 1, \cdots, n$, such that $x \in \bigcap_{i=1}^n f_i^{-1}(U_i)$. This is equivalent to for every $x \in V$, there exist $n \in \mathbb{N}$, $f_1, f_2, \cdots, f_n \in \mathscr{F}$, and open neighborhoods U_i of $f_i(x)$ in Y_{f_i} for $i = 1, \cdots, n$, such that

$$\{y \in X : f_i(y) \in U_i, i = 1, \cdots, n\} \subset V. \tag{4.2}$$

Remark 4.3 (Universality Property)**.** If Z is a topological space, then $g : Z \to X$ is continuous if and only if $g^{-1}(f^{-1}(U))$ is open in Z, for any $f \in \mathscr{F}$ and U is open in Y_f, which is equivalent to say $f \circ g : Z \to Y_f$ is continuous for any $f \in \mathscr{F}$.

Exercise 4.1. Show that if τ is a topology on X such that for any Z and $g : Z \to X$, g is continuous with respect to $\tau \Leftrightarrow f \circ g : Z \to Y_f$ is continuous for any $f \in \mathscr{F}$, then $\tau = \sigma(X, \mathscr{F})$.

Remark 4.4. If Y_f is Hausdorff for any $y \in \mathscr{F}$, and $\mathscr{F}$ separates the points of X (for any $x \neq y$, there exists a f such that $f(x) \neq f(y)$), then $\sigma(X, \mathscr{F})$ is Hausdorff.

Example 4.1 (subspace topology)**.** Let X be a topological space, $Y \subset X$ is a subspace, and $i : X \to Y$ be the inclusion map. Let τ be the topology of X, then $\sigma(Y, \{i\})$ is the **subspace topology of** Y, which is denoted by $\tau|_Y$.

Example 4.2 (product topology)**.** Let $X_\gamma, \gamma \in \Gamma$ be a family of topological spaces. Let $X = \prod_{\gamma\in\Gamma} X_\gamma$=$\{x$: x is a function on Γ such that $x(\gamma) = x_\gamma \in X_\gamma$, $\forall \gamma \in \Gamma\}$. X is the set of "Γ-tuples" $x = (x_\gamma)_{\gamma\in\Gamma}$. We have the projections $\pi_\delta : X \to X_\delta$ ($\delta \in \Gamma$), where $\pi_\delta(x) = x(\delta) = x_\delta$ for all $x = (x_\gamma)_{\gamma\in\Gamma}$.
The **product topology** on X is $\sigma(X, \{\pi_\gamma : \gamma \in \Gamma\})$. $V \subset X$ is open means that for every $x = (x_\gamma)_{\gamma\in\Gamma} \in V$, there exist $n \in \mathbb{N}$, $\gamma_1, \gamma_2, \cdots, \gamma_n \in \Gamma$, and open neighborhoods U_i of x_{γ_i} in X_{γ_i} for $i = 1, \cdots, n$, such that

$$\{y = (y_\gamma)_{\gamma\in\Gamma} \in X : y_{\gamma_i} \in U_i, i = 1, \cdots, n\} \subset V. \tag{4.3}$$

Proposition 4.1. *For each $n \in \mathbb{N}$, let (Y_n, d_n) be a metric space. Let X be a set, $f_n : X \to Y_n$ be functions that separate the points of X, then $\sigma(X, \{f_n|n \in \mathbb{N}\})$ is metrizable.*

Proof. If d is a metric, then so is $\frac{d}{d+1}$ which is equivalent to d. Without loss of generality, let's assume $d_n \leq 1$ for every $n \in \mathbb{N}$. Then define

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(f_n(x), f_n(y)), \tag{4.4}$$

which is a metric on X. We need show the topology generated by d is equivalent to $\sigma(X, \{f_n|n \in \mathbb{N}\})$. First assume $d(x, x_k)$ as $k \to \infty$, then $2^{-n} d_n(f_n(x), f_n(x_k)) \leq d(x, x_k)$ for every $n \geq 1$ Id: $(X, d) \to \sigma(X, \{f_n|n \in \mathbb{N}\})$ is continues. (use the universality property) Id: $\sigma(X, \{f_n|n \in \mathbb{N}\}) \to (X, d)$ is also continues. (by direct argument) □

Theorem 4.2 (Tychonov)**.** *The product of compact spaces is compact in product topology.*

4.2 Weak topologies on vector spaces

Let E be a real or complex vector space. Let F be a vector space of linear functionals on E such that separates the points of E (for every $x \neq 0$ in E, there exist an $f \in F$ such that $f(x) \neq 0$). We consider the weak topology $\sigma(E, F)$ on E. $U \subset E$ is open $\Leftrightarrow$ for every $x \in U$, there exist $n \in \mathbb{N}$, $f_1, \cdots, f_n \in F$, $\epsilon > 0$, such that

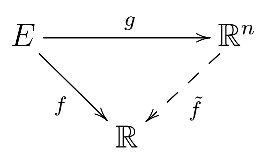
$$\{y \in E : |f_i(x) - f_i(y)| < \epsilon, i = 1, \cdots, n\} \subset U. \tag{4.5}$$

Remark 4.5. $(E, \sigma(E, F))$ is a locally convex space with defining seminorms: $x \to |f(x)|$ for $x \in E$ and $f \in F$.

Note. $(E, \sigma(E, F))$ is Hausdorff, and its addition and scalar multiplication are continuous.

Lemma 4.3. *Let E be as above, and let $f, g_1, \cdots, g_n$ be linear functionals on E. If $\ker f \supset \bigcap_{i=1}^n \ker g_i$, then $f \in \operatorname{Span}\{g_1, \cdots, g_n\}$.*

Proof. Define $g(x) = \big(g_1(x), g_2(x), \cdots, g_n(x)\big)$, with $\ker g \subset \ker f$ and $x \in E$. Then there exists a unique linear functional $\tilde{f} : g(E) \to \mathbb{R}$ such that $\tilde{f} \circ g = f$. Extend $\tilde{f}$ to the whole $\mathbb{R}^n$, then we can find a $b = (b_1, \cdots, b_n) \in \mathbb{R}^n$ such that $\tilde{f}\big((a_1, \cdots, a_n)\big) = \sum_{i=1}^n a_i b_i$. So $f(x) = \tilde{f} \circ g(x) = \sum_{i=1}^n b_i g_i(x)$, and f is therefore in the span of g_i's.



□

Proposition 4.4. *Let E, F be as above, then a linear functional $f : E \to \mathbb{R}$ is $\sigma(E, F)$-continuous if and only if $f \in F$, i.e. $(E, \sigma(E, F))^* = F$.*

Proof. ($\Leftarrow$) By definition.
($\Rightarrow$) Suppose $f : E \to \mathbb{R}$ is continuous in the $\sigma(E, F)$ topology. There exists an open neighborhood U of 0 in E such that $|f(x)| < 1$ for all $x \in U$. WLOG, let

$$U = \{x \in E : |g(x)| < \epsilon, i = 1, \cdots, n\} \tag{4.6}$$

for some $n \in \mathbb{N}$, $g_1, \cdots, g_n \in F$, and $\epsilon > 0$. Now if $x \in \bigcap_{i=1}^n \ker g_i$, then $\lambda x \in U$ for any scalars. Hence

$$|f(\lambda x)| = |\lambda||f(x)| < 1 \tag{4.7}$$

for any scalar λ, which implies that $f(x)$=0. Then $\bigcap_{i=1}^n \ker g_i \subset \ker f$, by previous lemma we have $f \in \mathit{span}\{g_1, \cdots, g_n\} \subset F$. □

Recall that we always identify the image of a normed space X under the canonical embedding $X \to X^{**}$ with X.

$$X \hookrightarrow X^{**}$$

Let X be a normed space.

Definition 4.2. Let $E = X, F = X^*$. Notice that by Hahn Banach theorem, X^* separates the points in X. Then $\sigma(X, X^*)$ is **the weak topology** of X. We write (X, w) for $(X, \sigma(X, X^*))$. Then $U \subset X$ is **weakly open** (or w-open), i.e. $U \in \sigma(X, X^*) \iff \forall x \in U, \exists \epsilon > 0, \exists n \in \mathbb{N}, \exists x_1^*, \cdots, x_n^* \in X^*$ such that

$$\{y \in X : |x_i^*(y) - x_i^*(x)| < \epsilon, i = 1, \cdots, n\} \subset U. \tag{4.8}$$

Definition 4.3. Let $E = X^*, F = X \hookrightarrow X^{**}$. Then $\sigma(X^*, X)$ is **the weak-star topology** of X^*. We write (X^*, w^*) for $(X^*, \sigma(X^*, X))$. Then $U \subset X$ is **weak-* open** (or w*-open), i.e. $U \in \sigma(X^*, X) \iff \forall x^* \in U, \exists \epsilon > 0, \exists n \in \mathbb{N}, \exists x_1, \cdots, x_n \in X$ such that

$$\{y^* \in X^* : |y^*(x_i) - x^*(x_i)| < \epsilon, i = 1, \cdots, n\} \subset U. \tag{4.9}$$

Hence last proposition directly gives

Proposition 4.5. *A linear functional $f : X \to \mathbb{R}$ is w-continuous $\Leftrightarrow f \in X^*$. Similarly $g : X^* \to \mathbb{R}$ is w*-continuous $\Leftrightarrow g \in X$. i.e. $(X, w)^* = X^*$, $(X^*, w^*)^* = X$.*

It follows that $\sigma(X^*, X^{**}) = \sigma(X^*, X)$ if and only if X is reflexive.

Properties

- (X, w) and (X^*, w^*) are locally convex spaces, hence Hausdorff. In addition, the scalar multiplications are continuous.
- $\sigma(X, X^*) \subset \|\cdot\|$ topology, i.e. the weak topology of X is a subset of the topology on X induced by norm. Similarly we have $\sigma(X^*, X) \subset \sigma(X^*, X^{**}) \subset \|\cdot\|$ topology (on X^*).
- If $\dim X < \infty$, then all these topologies coincide.
- If $\dim X = \infty$, and U is a w-open neighborhood of 0, then U is not bounded in norm. Hence $\sigma(X, X^*) \subsetneq \|\cdot\|$ topology. Moreover, (X, w) is not metrizable (not even first countable).
- If $\dim x$ is uncountable (e.g. X is complete and $\dim x = \infty$), then (X^*, w^*) is not metrizable (not even first countable).
- Let Y be a subspace of X, then $\sigma(X, X^*)|_Y = \sigma(Y, Y^*)$ (by Hahn Banach). Similarly $\sigma(X^{**}, X^*)|_X = \sigma(X, X^*)$. So the canonical embedding $X \to X^{**}$ is a weak-to-weak-* homeomorphism into X^{**}.

4.3 Weak and weak-* convergence

In X, $x_n \xrightarrow{w} x$ means that $\{x_n\}$ **converges weakly** (i.e. in the weak topology) to x. This is equivalent to

$$< x_n, x^* > \longrightarrow < x, x^* > \tag{4.10}$$

for any $x^* \in X^*$.

Similarly in X^*, $x_n^* \xrightarrow{w^*} x^*$ means that $\{x_n^*\}$ **converges w-*** (i.e. in the weak-star topology) to x^*. This is equivalent to

$$< x, x_n^* > \longrightarrow < x, x^* > \tag{4.11}$$

for all $x \in X$.

Definition 4.4. $B \subset X^*$ is said to be weakly bounded if $\{x^*(x) : x^* \in B\}$ is bounded $\forall x \in X$.

Remark 4.6. The **principle of uniform boundedness** (PUB) says that:
let X be a Banach space, Y be a normed space, and $\mathcal{T} \in \mathcal{B}(X,Y)$ be a collection of linear maps which is also pointwise bounded (i.e. $\sup_{T\in\mathcal{T}} \|T(x)\| < \infty$ for any $x \in X$). Then $\mathcal{T}$ is uniformly bounded, that is, $\sup_{T\in\mathcal{T}} \|T\| < \infty$.

Proposition 4.6.

- *Let X be a normed space, and $A \subset X$ be weakly bounded, then A is norm-bounded.*
- *Let X be a Banach space, and $B \subset X^*$ be weak-* bounded, then B is norm-bounded.*

Proof. i) Since $A \subset X \subset X^{**} = \mathcal{B}(X^*, \mathbb{R})$, A is weakly bounded is equivalent to A is pointwise bounded. In addition, X^* is complete. Hence the result follows from PUB.
ii) Notice $B \subset X^* = \mathcal{B}(X, \mathbb{R})$ which means that B is w-* bounded $\leftrightarrow$ B is pointwise bounded. Since X is complete, we can apply PUB again. □

Proposition 4.7.

- *Let X be a normed space. If $x_n \xrightarrow{w} x$, then $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$ and $\|x\| \le \liminf \|x_n\|$.*
- *Let X be a Banach space. If $x_n^* \xrightarrow{w^*} x^*$, then $\sup_{n\in\mathbb{N}} \|x_n^*\| < \infty$ and $\|x^*\| \le \liminf \|x_n^*\|$.*

Proof. i) Since $x^*(x_n) \to x*(x)$ for every $x^* \in X^*$, $\{x^*(x_n) : n \in \mathbb{N}\}$ is bounded. Hence $\sup_{n\in\mathbb{N}} \|x^*(x_n)\| < \infty$ and the result follows from the previous proposition.

$$|x^*(x)| = \liminf_{n\to\infty} |x^*(x_n)| \le \liminf_{n\to\infty} \|x^*\| \cdot \|x_n\|. \tag{4.12}$$

Pick $x^* \in X^*$ such that $\|x^*\| = 1$ and $x^*(x) = \|x\|$. We obtain that $\|x\| \le \liminf \|x_n\|$.

ii) Similar to i). □

4.4 Hahn Banach separation theorem

Let $(X, \mathcal{P})$ be a locally convex space. Suppose C is a convex subset of X with $0 \in int\ C$. We define $\mu_C : X \to \mathbb{R}$ by

$$\mu_C = \inf\{t > 0 : x \in tC\}. \tag{4.13}$$

For $x \in X$, $0 \cdot x = 0 \in int\ C$, then by the continuity of scalar multiplication, there exists some $\delta > 0$ such that for any scalar λ such that $|\lambda| < \delta$, we have $\lambda x \in C$. Therefore $x \in \frac{1}{\delta} C$. So μ_C is well-defined.

Example 4.3. Let X be a normed space, and $C = B_x$ is the unit ball. Then $\mu_C = \|\cdot\|$.

Lemma 4.8. *μ_C is positive homogeneous and subadditive.*

$$\{x \in X : \mu_C(x) < 1\} \subset C \subset \{x \in X : \mu_C(x) \leq 1\}. \tag{4.14}$$

Furthermore, if C is open, then

$$C = \{x \in X : \mu_C(x) < 1\}. \tag{4.15}$$

Proof. From the definition, we get $\mu_C(tx) = t\mu_C(x)$, $\forall x \in X$, $\forall t > 0$. Also $\mu_C(0) = 0$.
Now given $x, y \in X$, fix $s > \mu_C(x)$, $t > \mu_C(y)$. So there exist $s' < s$ such that $x \in s'C$. Then

$$\frac{x}{s} = \frac{s'}{s} \cdot \frac{x}{s'} + (1 - \frac{s'}{s}) \cdot 0 \in C \tag{4.16}$$

since C is convex. So $x \in sC$. Similarly, $y \in tC$. It follows that

$$\frac{s}{s+t} \cdot \frac{x}{s} + \frac{t}{s+t} \cdot \frac{y}{t} = \frac{x+y}{s+t} \in C. \tag{4.17}$$

Hence $x + y \in (s+t)C$ and $\mu_C(x+y) \leq s+t$. Taking infimum over all s, t, we get $\mu_C(x+y) \leq \mu_C(x) + \mu_C(y)$.
For the second part, note that $\mu_C(x) < 1 \Rightarrow x \in C$ is shown by above argument. $x \in C \Rightarrow \mu_C(x) < 1$ is by definition.
Finally, suppose C is open and $x \in C$, then $x \cdot 1 \in C$. By the continuity of scalar multiplication, there exists some $\delta > 0$ such that $(1+\delta)x \in C$. Therefore, $\mu_C(x) \leq 1/(1+\delta) < 1$. □

Remark 4.7. C is called **symmetric** if $x \in C$ implies $-x \in C$. C is called **balanced** if $x \in C$, $\lambda \in \mathbb{C}$, and $|\lambda| = 1$ implies $\lambda x \in C$. Note that in the case of real scalars, "balanced"="symmetric".

Remark 4.8. If U is a neighborhood of 0, then there exists a convex and balanced neighborhood of 0 such that $V \subset U$. Indeed, there exist $n \in \mathbb{N}$, $p_1, \cdots, p_n \in \mathcal{P}$, and some $\epsilon > 0$, such that

$$V = \{x \in X : p_i(x) < \epsilon, i = 1, \cdots, n\} \subset U. \tag{4.18}$$

Remark 4.9. If U is a neighborhood of 0, then there exists a convex and balanced neighborhood of 0 such that $V + V \subset U$. By previous remark, we can assume V to be convex and balanced.

Theorem 4.9. *(Hahn-Banach separation theorem) Let $(X, \mathcal{P})$ be a real or complex locally convex space. Let C be an open convex set in X such that $0 \in C$. Given $x_0 \in X \setminus C$, there exists an $f \in X^*$ such that $f(x) < f(x_0)$ for all $x \in C$. (In complex case, $\Re f(x) < \Re f(x_0)$, $\forall x \in C$.)*

Proof. i) (Real case) By previous lemma, we have a positive homogeneous and subadditive functional μ_C. We define $f : span\{x_0\} \to \mathbb{R}$ by setting $f(tx_0) = t\mu_C(x_0)$. For $t \geq 0$

$$f(tx_0) = t\mu_C(x_0) = \mu_C(tx_0). \tag{4.19}$$

For $t < 0$,

$$f(tx_0) = t\mu_C(x_0) \leq 0 \leq \mu_C(tx_0). \tag{4.20}$$

In *span*$\{x_0\}$, f is dominated by μ_C, so we can extend f to the whole X and is still dominated by μ_C (Hahn-Banach).
If $x \in C$, then $f(x) \leq \mu_C(x) < 1 \leq \mu_C(x_0) = f(x_0)$ by using lemma 4.8. Since C is a neighborhood of 0, there exists a symmetric neighborhood U of 0 such that $U \subset C$. For $x \in U$, $\pm x \in U \subset C$. Hence $\pm f(x) < 1$, i.e. $|f(x)| < 1$, which implies f is continuous at 0, and so $f \in X^*$ by lemma 2.9.

ii) (Complex case) Consider X as a real vector space, by the first part there exists a real continuous linear functional g on X such that $g(x) < g(x_0)$ for any $x \in C$. Setting $f(x) = g(x) - ig(ix)$, $x \in X$ and we have $\Re f = g$. □

Remark 4.10. From now on, we only state and prove the real version since complex version follows similarly as in ii).

Theorem 4.10 (Hahn-Banach separation theorem)**.** *Let $(X, \mathcal{P})$ be a locally convex space. Let A, B be disjoint non-empty open convex sets of X.*

i) If A is open, there exist $f \in x^*$, $\alpha \in \mathbb{R}$ such that $f(a) < \alpha \leq f(b)$, $\forall a \in A, b \in B$.

ii) If A is compact, B is closed, then there exists an $f \in X^*$ such that $\sup_A f < \inf_B f$.

Proof. i) Fix $a_0 \in A$, $b_0 \in B$. Let $x_0 = a_0 - b_0$, $C = A - B + x_0 = \bigcup_{b\in B}(A - b + x_0)$. Then C is an open convex set and $0 \in C$. Since $A \cap B = \emptyset$, we have $x_0 \notin C$. Then by Theorem 4.9, we can find an $f \in X^*$ such that $f(x) < f(x_0)$ $\forall x \in C$, i.e.

$$f(a - b + x_0) < f(x_0), \quad \forall a \in A, b \in B. \tag{4.21}$$

So $f(a) < f(b)$ for all $a \in A$ and $b \in B$. It follows that $\alpha = \inf_B f$ exists. Since $f(a_0) < f(b_0)$, we have $f \neq 0$. Pick any $z \in X$ such that $f(z) > 0$. Given any $a \in A$, as A is open, there exists a $\delta > 0$ such that $(a + \delta z) \in A$. Hence, $f(a) < f(a + \delta z) \leq \alpha$.

ii) For any $a \in A$, there exists an open neighborhood U_a of 0 such that $(a + U_a) \cap B = \emptyset$ (since B is closed). There exists a balanced convex open neighborhood V_a of 0 such that $V_a + V_a \subset U_a$. $\{a + V_a\}_{a\in A}$ is an open cover for A, so there exists $a_1, \cdots, a_n \in A$, $n \in \mathbb{N}$ such that $A = \bigcup_{i=1}^n (a_i + V_{a_j})$. Define $V = \bigcap_{i=1}^n V_{a_i}$ which is an balanced convex open neighborhood of 0, and we have $(A + V) \cap B = \emptyset$. Let $a \in A$ be arbitrary, then there exists a j such that $a \in (a_j + V_{a_j})$, so that $(a + V) \in (a_j + V_{a_j} + V) \subset (a_j + U_{a_j})$ and $(a_j + U_{a_j}) \cap B = \emptyset$. Hence $A + V$ is an open convex set, and by i) there exist $f \in X^*, \beta \in \mathbb{R}$ such that $f(a + v) < \beta \leq f(b)$, $\forall a \in A$, $v \in V$, and $b \in B$.
In particular, $f \neq 0$, so there exists a $z \in V$ such that $f(z) > 0$. Hence $f(a) < \beta - f(z)$ for all $x \in A$. Therefore, $\alpha = \sup_A f < \beta$. (Or $f(a) < \beta$, $\forall a \in A$, and $\sup_A f$ is attained.) □

Theorem 4.11 (Mazur)**.** *Let X be a normed space, and C be a convex set in X. Then C is weakly closed if and only if C is norm-closed. Hence for general convex sets C , we have $\overline{C}^w = \overline{C}^{\|\cdot\|}$.*

Proof. (⇒) Clear.
(⇐) Let $x \in X \setminus C$. Apply theorem 4.10 ii) to $A = \{x\}$, B=C, and $\mathcal{P} = \|\cdot\|$. So there exists an $f \in x^*$ such that $f(x) \leq \inf_C f = \alpha$. $\{z \in X : f(z) < \alpha\}$ is a weakly open set containing x, which is disjoint from C. Thus $X \setminus C$ is weakly open and then C is weakly closed. □

Corollary 4.12. *If $x_n \xrightarrow{w} 0$ in a normed space X, then for any $\epsilon > 0$, there exist $n \in \mathbb{N}$, $t_i \geq 0$ for $i = 1, \cdots, N$, and $\sum_{i=1}^N t_i = 1$, such that $\| \sum_{i=1}^N t_i x_i \| < \epsilon$.*

Proof. Let $C = conv\{x_i : i \in \mathbb{N}\} = \{\sum_{i=1}^n t_i x_i : n \in \mathbb{N}, t_i \geq 0, \forall \sum_{i=1}^n t_i = 1\}$. As $x_n \xrightarrow{w} 0$, $0 \in \overline{C}^w = \overline{C}^{\|\cdot\|}$ by Theorem 4.11. □

Theorem 4.13 (Banach-Alaoglu)**.** *In any normed space X, (B_{X^*}, x^*) is compact.*

Proof. For $x \in X$, let $K_x = \{\lambda : |\lambda| \leq \|x\|, \lambda \text{ is a scalar}\}$. Set $K = \prod_{x \in X} K_x$ with the product topology. This set is compact by Tychonov theorem. Note that

$$K = \{f : X \to \text{scalars} : |f(x)| \leq \|x\|\}. \tag{4.22}$$

So $B_{X^*} \subset K$ and $B_{X^*} = \{f \in K : f \text{ is linear}\}$. The product topology on K is the smallest topology on K such that $\pi_x : K \to K_x$ is continuous for every $x \in X$. Note that $\pi_x(f) = f(x)$. The weak-* topology on X^* is the smallest topology on X^* such that $\hat{x}|_{B_{X^*}}$ is continuous for every $x \in X$ (here we use the identification $X \hat{\to} X^{**}$, with $\hat{x}|_{B_{X^*}} = f(x)$. Hence (B_{X^*}, w^*) is a subspace of K, and it suffices to show B_{X^*} is closed in K. But

$$\begin{aligned} B_{X^*} =& \{f \in K : f(\lambda x + \mu y) - \lambda f(x) - \mu f(y) = 0, \forall x, y \in X, \lambda, \mu \text{ are scalars}\} & (4.23)\\ =& \bigcap_{\substack{x,y \in X \\ \lambda,\, \mu \text{ are scalars}}} \{f \in K : \big(\pi_{\lambda x + \mu y} - \pi_\lambda - \pi_\mu\big)(f) = 0\} & (4.24) \end{aligned}$$

which is clearly closed. □

Proposition 4.14. *Let X be a normed space, and K be a compact Hausdorff space, then*

1. *X is separable ⇔ (B_{X^*}, w^*) is metrizable.*
2. *$C(K)$ is separable ⇔ K is metrizable.*

Proof. 1. (⇒) Choose a dense subset $\{x_n : n \in \mathbb{N}\} \subset X$. Let $\sigma = \sigma\big(B_{X^*}, \{\hat{x}|_{B_{X^*}} : n \in \mathbb{N}\}\big)$, which is the smallest topology on B_{X^*} such that $x^* \mapsto x^*(x_n)$ is continuous for any $n \in \mathbb{N}$. So $\sigma \subset$ w-* topology of B_{X^*}, which implies that the formal identity

$$i : (B_{X^*}, w^*) \to (B_{X^*}, \sigma) \tag{4.25}$$

is continuous. Since $\{x_n : n \in \mathbb{N}\}$ is dense in X, they separate the points of B_{X^*}. By Proposition 4.1, (B_{X^*}, σ) is metrizable. Moreover, i is a continuous bijection from a compact space to a Hausdorff space, hence it is a homeomorphism.

2. (⇒) Let $X = C(K)$ be separable, then by above result we see that (B_{X^*}, w^*) is metrizable. Define $\delta : K \to (B_{X^*}, w^*)$ which maps k to δ_k for $k \in K$, where $\delta_k(f) = f(k)$ for $f \in C(K)$.

δ is continuous since $k \mapsto \delta_k(f) = f(k)$ is continuous for every $f \in C(K)$. Since K is compact and Hausdorff, it is also normal. By Uryson's lemma, for any $k \neq k'$ in K, there exists an $f \in C(K)$ such that $f(k) \neq f(k')$, thus $\delta_k \neq \delta_k$ if $k \neq k'$. Therefore, $\delta : K \to \delta(K)$ is a continuous bijection from a compact space to a Hausdorff space. Then we see that K is homeomorphic to its image in the metric space (B_{X^*}, w^*) which implies K is metrizable.

2. ($\Leftarrow$) Since K is a compact metric space, K is separable. Let $\{k_n : n \in \mathbb{N}\}$ be a dense set in X. Define $f_0 = 1$, $f_n(k) = d(k, k_n)$ for $k \in K$ and $n \geq 1$. Let A be the algebra generated by f_n, $n \geq 0$, that is,

$$A = span\{\prod_{n\in F} : F \text{ is a finite subset of } \{0, 1, 2, \cdots\}\}. \tag{4.26}$$

Then A is separable, $1 \in A$, and A separates the points of K: $\forall k \neq k' \in K$, $\exists n \in \mathbb{N}$ such that $d(k, k_n) < d(k', k_n)$. By Stone-Weierstrass theorem, $\bar{A} = C(K)$, which implies that $C(K)$ is separable.

1. ($\Leftarrow$) Let $K = (B_{X^*}, w^*)$, then by part 2, $C(K)$ is separable. Consider $X \subset C(K)$ with the identification $x \mapsto \hat{x}\big|_K$ defined by $\hat{x}\big|_K(x^*) = x^*(x)$. This is well defined

$$\|\hat{x}\big|_K\|_\infty = \sup\{|x^*(x) : x^* \in B_{X^*}\} = \|x\|. \tag{4.27}$$

by Hahn-Banach theorem. Hence X is separable. □

Remark 4.11. X is separable $\Rightarrow$ X^* is w-* separable, and $X^* = \cup_{n=1}^{\infty} bB_{X^*}$. ("$\Leftarrow$ is false in general, e.g. $X = l^\infty$")

Remark 4.12. X is separable $\Rightarrow$ X is w-separable. (If $A \subset X$, then $\overline{span}A = \overline{span}^w A) \supset \bar{A}^w \supset \bar{A}$)

Proposition 4.15. *Let X be a normed space. X^* is separable if and only if (B_X, w) is metrizable.*

Proof. ($\Rightarrow$)By previous proposition, $(B_{X^{**}}, w^*) = (B_{X^{**}}, \sigma(X^{**}, X^*))$ is metrizable. Since (B_x, w) is a subspace of $(B_{X^{**}}, w^*)$, it is metrizable.
($\Leftarrow$) Assume that (B_x, w) is metrizable by metric d. For any weakly open neighborhood U of 0, there exists an $n \in \mathbb{N}$ such that $B(0, \frac{1}{n}) = \{x \in B_X : d(x, 0) < \frac{1}{n}\} \subset U$. For every n, there exist a finite set $F_n \subset X^*$, $\epsilon_n > 0$, such that $U_n = \{x \in B_X : |X^*(x)| < \epsilon_n, \forall x^* \in F_n\}$, and $U_n \subset B(0, \frac{1}{N})$. Let $Z = \overline{span} \cup_{n\in\mathbb{N}} F_n$, then Z is separable. We will show $Z = X^*$.
Suppose not, then there exists an $x^* \in B_{X^*}$ with $d(x^*, Z) = \inf_{z\in Z} \|x^* - z\| > 1/2$. Then there exists an $n \in \mathbb{N}$ such that $U_n \subset \{x \in B_X : |x^*(x)| < 1/10\}$ since $\{x \in B_X : |x^*(x)| < 1/10\}$ is a weakly open neighborhood of 0 in B_X. Now let $Y = \cap_{y^*\in F_n} \ker y^*$. If $y \in B_Y$, then $y \in U_n$ since $|x^*(y)| < 1/10$. So $\|x^*|_Y\| \leq 1/10$. By Hahn-Banach theorem, there exists a $z^* \in X^*$ such that $\|z^*\| \leq 1/10$ and $z * |_Y = x^*|_Y$. Since

$$Y = \cap_{y^*\in F_n} \ker y^* \subset \ker(x^* - z^*). \tag{4.28}$$

By lemma 4.3, $(x^* - z^*) \subset span_{n\in\mathbb{N}} F_n \subset Z$. Thus, $d(x^*, Z) \leq 1/10$ which gives us a contradiction. □

Proposition 4.16. *Let X be a normed space, $K \subset X$ and (K, w) is compact. If X^* is w-* separable, then (K, w) is metrizable.*

Note. If X is separable, then X^* is w-* separable.

Proof. Let A be a countable subset of X^* such that $\overline{A}^{w^*} = A^*$. Then A separates the points of X. By proposition 4.1, $\sigma = \sigma(K, A)$ is metrizable. Since $A \subset X^*$, the formal identity

$$(K, w) \to (K, \sigma) \tag{4.29}$$

is a continuous bijection from a compact space to a Hausdorff space, hence is a homeomorphism. □

Theorem 4.17 (Goldstein)**.** *Let X be a normed space, then $\overline{B_X}^{w^*} = B_{X^{**}}$. Here we view B_X sitting inside $B_{X^{**}}$.*

Proof. Let $K = \overline{B_X}^{w^*}$. Since $B_X \subset B_{X^{**}}$ and $B_{X^{**}}$ is w-* closed, we have $K \subset B_{X^{**}}$. Suppose $K \neq B_{X^{**}}$. Pick $x^{**} \in B_{X^{**}} \setminus K$. It is easy to check that K is compact. By theorem 4.10 (ii), there exists a w-* continuous linear functional $x^* \in X^*$ such that

$$\sup_{z^{**}\in K} z^{**}(x^*) < x^{**}(x^*). \tag{4.30}$$

Since $K \supset B_X$,

$$\sup_{z^{**}\in K} z^{**}(x^*) \geq \sup_{z^{**}\in B_X} x^*(x) = \|x^*\|. \tag{4.31}$$

But

$$x^{**}(x^*) \leq \|x^{**}\| \cdot \|x^*\| \leq \|x^*\| \tag{4.32}$$

which gives a contradiction. □

Theorem 4.18. *Let X be a Banach space, TFAE:*

1. *X is reflexive.*
2. *(B_X, w) is compact.*
3. *X^* is reflexive.*

Proof. 1. ⇒ 2. Since X is reflexive, $X = X^{**}$, which implied that the weak topology on X is the same as the w-* topology on X^{**}. Then $(B_X, w) = (B_{X^{**}}, w^*)$ is compact by Banach-Alaoglu theorem.

2. ⇒ 1. The restriction of the w-* topology on X^{**} to X is the weak topology on X. Since B_X is weakly compact, it is a w-* compact subset of $B_{X^{**}}$ hence is w-* closed. By Goldstein's theorem, $B_{X^{**}} = \overline{B_X}^{w^*} = B_X$, which implies that $X = X^{**}$.

1. $\Rightarrow$ 3. If X is reflexive, $\sigma(X^*, X) = \sigma(X^*, X^{**})$. So $(B_{X^*}, w) = (B_{X^*}, w^*)$, which is compact by Banach-Alaoglu theorem. Then by "2. $\Rightarrow$ 1.", X^* is reflexive.

3. $\Rightarrow$ 1. If X^* is reflexive, $\sigma(X^{**}, X^*) = \sigma(X^*, X^{***})$. B_X is norm-closed in X^{**} since X is complete. So B_X is weakly closed by Mazur's theorem. Then B_X is w-* closed in X^{**}. By Goldstein's theorem, $B_{X^{**}} = \overline{B_X}^{w^*} = B_X$ and X is then reflexive.

$\square$

Remark 4.13. 1. $\Leftrightarrow$ 3. has a easy direct proof.

Remark 4.14. If X is reflexive and separable, then (B_{X^*}, w) is a compact metric space.

Recall that we have shown if X is separable, then $X \hookrightarrow l^\infty$ isometrically. Now we aim to show that $X \hookrightarrow C[0,1]$ isometrically.

Lemma 4.19. *If K is a non-empty compact metric space, then K is a continuous image of the Cantor set $\triangle$. Here $\triangle = \{0,1\}^{\mathbb{N}}$ with the product topology. Note that $\triangle$ is a compact metric space by proposition 4.1 and theorem 4.2.*

Note. $\triangle$ is homeomorphic to

$$\{\sum_{n=1}^{\infty}(2\epsilon_n)3^{-n} : (\epsilon_n)_{n=1}^{\infty} \in \triangle\} \subset C[0,1]. \tag{4.33}$$

Theorem 4.20. *Let X be a normed space. If X is separable, then $X \hookrightarrow C[0,1]$ isometrically.*

Proof. Let $K = (B_{X^*}, w^*)$, then K is a compact metric space. the map $X \to C(K) : x \mapsto \hat{x}|_K$ is an isomorphism into $C(K)$. By lemma 4.19, there exists a continuous surjective map $\phi : \triangle \to K$, which yields an isometric isomorphism $C(K) \to C(\triangle) : f \mapsto f \circ \phi$ into $C(\triangle)$, where $f \in C(K)$
Finally, we have an isometric isomorphism $C(\triangle) \to C[0,1]$: given $g \in C[0,1]$, thinking of $\triangle \subset C[0,1]$, we extend g to the whole $[0,1]$ to a piecewise linear function. $\square$

5 The Krein-Milman theorem

6 Banach algebras

6.1 Elementary properties and examples

Let A be an algebra over $\mathbb{R}$ or $\mathbb{C}$, i.e. a vector space with multiplication which satisfies

- $(ab)c = a(bc)$;
- $a(b+c) = ab + ac$;
- $(a+b)c = ac + bc$;
- $\lambda(ab) = \lambda(ab)$;

for any scalar λ. An algebra A is both a ring and a vector space. The structure $(A, \cdot)$ is a semigroup. An algebra is **commutative** is its ring multiplication is commutative.

Definition 6.1. An (two sided) **ideal** I of an algebra A is a subset of A such that

- I is a vector subspace of A;
- $AI \subset I$ and $IA \subset I$.

Definition 6.2. An algebra norm $\|\cdot\|$ on A is a norm such that

$$\|ab\| \leq \|a\| \cdot \|b\| \tag{6.1}$$

for all $a, b \in A$. The pair $(A, \|\cdot\|)$ is a **normed algebra**.

Note. The multiplication is continuous: $a_n \to a, b_n \to b$ implies $a_n b_n \to ab$.

Definition 6.3. A **Banach algebra**(B.A.) is a complete normed algebra.

An algebra A is **unital** if there exist elements 1 (or 1_A) such that $1 \neq 0$, $1a = a1 = a$, $\forall a \in A$. A **unital normed algebra** is a normed unital algebra such that $\|1\| = 1$. If $\|1\| \neq 1$, then one can find an equivalent norm $|||\cdot|||$ such that $|||1||| = 1$, for instance,

$$|||a||| = \sup\{\|ba\| : b \in A, \|b\| \leq 1\}. \tag{6.2}$$

Definition 6.4. A **unital Banach algebra** is a complete unital normed algebra.

A **homomorphism** between algebras A, B is a linear map $\Phi : A \to B$ such that $\Phi(ab) = \Phi(a)\Phi(b)$, for all $a, b \in A$. If A, B are unital, then we say Φ is unital if $\Phi(1_A) = 1_B$.
If A, B are normed algebras, a homomorphism $\Phi : A \to B$ may or may not be continuous. However, by an **isomorphism** we mean a bijective homomorphism $\Phi : A \to B$ such that both Φ and Φ^{-1} are continuous.
From now on we assume our scalar field to be $\mathbb{C}$.

Example 6.1. Let K be a compact, Hausdorff topological space. Then $C(K)$ is a unital, commutative Banach algebra with pointwise multiplication and sup-norm.

Example 6.2. The **uniform algebras** are closed subalgebra of $C(K)$, which contain 1 and separate the points of K. For example, let $K = \Delta\{z \in \mathbb{C} : |z| \leq 1\}$ the **disc algebra**, then

$$A(\Delta) = \{f \in C(\Delta) : \ f|_{int\Delta} \text{ is analytic}\} \tag{6.3}$$

is a uniform algebra. More generally, for a nonempty and compact space $K \subset \mathbb{C}$, we have

$$P(K) \subset R(K) \subset O(K) \subset A(K) \subset C(K), \tag{6.4}$$

where these are the closures in $C(K)$ of the subalgebra of, respectively, polynomial functions, ration functions without poles in K, functions that are analytic on an open neighborhood of K, and $A(K) = \{f \in C(K) : \ f|_{int\Delta} \text{ is analytic}\}$. We'll use the fact that

$$R(K) = P(K) \Longleftrightarrow \mathbb{C} \setminus K \text{ is connected} \tag{6.5}$$

$$A(K) = C(K) \Longleftrightarrow \text{int}K = \emptyset. \tag{6.6}$$

Example 6.3. Let K be $L^1(\mathbb{R})$ with convolution as multiplication

$$f * g = \int_{-\infty}^{\infty} f(s)g(s-t)ds, \tag{6.7}$$

then K is a non-unital commutative Banach algebra.

Example 6.4. Let X be a Banach space,

$$\mathcal{B}(X) = \{T : X \to X, T \text{ is linear and bounded}\}, \tag{6.8}$$

then $\mathcal{B}(X)$ is a unital Banach algebra under composition as multiplication and operator norm. If $\dim X > 1$, then $\mathcal{B}(X)$ is non-commutative.
An important special case: closed subalgebras of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. For example $\mathcal{B}(\ell_2^n) \cong M_n(\mathbb{C})$.

6.2 Elementary constructions

- Every closed subalgebra of a Banach algebra is a Banach algebra. A unital subalgebra of a unital algebra is a subalgebra containing the unit.

- *Unitization* Let A be a complex algebra. Let $A_+ = A \oplus \mathbb{C}$ with multiplication

  $$(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu). \tag{6.9}$$

  Then A_+ is a unital algebra with identity $1_{A_+} = (0, 1)$. If A is a normed algebra, then so is A_+ with norm $\|(x, \lambda)\| = \|x\| + |\lambda|$. Note that $\|1\| = 1$.

  A is identified with $\{(x, 0) : x \in A\}$ which is a closed ideal of A_+. A_+ is complete if and only if A is complete.

- *Ideals* If I is an ideal of a normed algebra A, then so is $\bar{I}$. If I is a closed ideal of A, then A/I is a normed algebra. If A is a Banach algebra. and I is a proper and closed ideal of A, then A/J is a unital normed algebra. ($\|1 + I\| = 1$ will follow from lemma 1 below.)

- *Completeion* Every normed algebra has a completion which is a Banach algebra. Let $X = \tilde{A}$ be the completion of A as a Banach space. For $a \in A$, $L_a(b) = ab$ for any $b \in A$. We can extend L_a uniquely to a bounded linear operator $\tilde{L_a}$ on X. It's easy to prove that $a \mapsto \tilde{L_a}$ is an isometric isomorhpism of A onto a subalgebra of Banach algebra $\mathcal{B}(X)$. Take the closure of that subalgebra in $\mathcal{B}(X)$ we can get a completion of A.

6.3 Group of units and spectrum

Lemma 6.1. *Let A be a unital Banach algebra and $x \in A$. If $\|1 - x\| < 1$, then x is invertible.*

Proof. Let $x = 1 - h$, where $h = 1 - x$. Note that $\|h\| \leq 1$.

$$\sum_{n=0}^{\infty} \|h^n\| \leq \sum_{n=1}^{\infty} \|h\|^n \leq \frac{1}{1-\|h\|} < \infty, \tag{6.10}$$

so $s = \sum_{n=1}^{\infty} h^n$ converges and $xs = (1-b)\sum_{n=1}^{\infty} h^n = 1$. Similarity we have $sx = 1$. □

For a unital algebra A, let $G(A)$ be the group of invertible elements of A.

Definition 6.5. Let A be a unital algebra, $x \in A$. The **spectrum** of x in A is $\sigma_A(x) = \sigma(x) = \{\lambda \in \mathbb{C} | \lambda 1 - x \notin G(A)\}$.

If A is non-unital, let $\sigma_A(x) = \sigma_{A+}(x)$.

Example 6.5. Let $A = M_n(\mathbb{C})$, then $\sigma_A(x) =$ set of eigenvalues of x.

Example 6.6. Let $A = C(K)$, where K is compact and Hausdorff, then $\sigma_A(f) = f(K)$.

Theorem 6.2. *If A is a Banach algebra, then $\sigma_A(x)$ is a non-empty, compact set of $\{\lambda \in C : \|\lambda\| \leq \|x\|\}$ for any $x \in A$.*

Proof. Without loss of generality, we can assume A is unital. Let $x \in A$, if $\lambda \in \mathbb{C}$ and $|\lambda| > \|x\|$, then $\|\frac{x}{\lambda}\| < 1$, so $1 - \frac{x}{\lambda} \in G(A)$ by lemma 1. So $\lambda 1 - x \in G(A)$ and $\lambda \notin G(A)$.
The function $\lambda \mapsto \lambda 1 - x : \mathbb{C} \mapsto A$ is continuous. $\sigma_A(x) =$ inverse image of the closed set $A \setminus G(A)$ by Corollary 2.
Define $f : \mathbb{C} \setminus \sigma_A(x) \mapsto A$ by $f(\lambda) = (\lambda 1 - x)^{-1}$.

$$\begin{aligned} f(\lambda) - f(\mu) =& (\lambda 1 - x)^{-1} - (\mu 1 - x)^{-1} && (6.11)\\ =& (\mu 1 - x)^{-1}[(\mu 1 - x) - (\lambda 1 - x)](\lambda 1 - x)^{-1} && (6.12)\\ =& (\mu - \lambda) f(\lambda) f(\mu). && (6.13) \end{aligned}$$

Hence

$$\frac{f(\lambda) - f(\mu)}{\lambda - \mu} = -f(\lambda)f(\mu) \tag{6.14}$$

which converges to $-f(\lambda)^2$ as $\mu \to \lambda$. Since f is continuous by Corollary 2, we get f is analytic.
If $|\lambda| > \|x\|$, then

$$\|(\lambda 1 - x)^{-1}\| = \|\frac{1}{\lambda}(1 - \frac{x}{\lambda})^{-1}\| \leq \frac{1}{|\lambda|} \cdot \frac{1}{1 - \frac{\|x\|}{|\lambda|}} = \frac{1}{|\lambda| - \|x\|} \tag{6.15}$$

which tends to 0 as $\lambda \to \infty$. (by lemma 1)
If $\sigma_A(x) = \emptyset$, then f is analytic on $\mathbb{C}$, bounded ($f(\lambda) \to 0 as |\lambda| \to \infty$). Hence by Liouville's theorem, f is a constant function and $f \equiv 0$, which gives a contradiction. □

Example 6.7. Let A be an algebra of complex valued functions on a set K. Assume A is a Banach algebra in some norm $\|\cdot\|$. For $f \in A, x \in K$, if $f(x) \neq 0$, then $f(x) \in \sigma_A(x)$. By Theorem 3, $|f(x)| \leq \|f\|$, so $A \subset l_\infty(K)$. Also, $\|f\|_\infty = \sup_K |f| \leq \|f\|$.

Corollary 6.3 (Gelfand-Mazur)**.** *If A is a unital normed complex division algebra, then $A \cong \mathbb{C}$.*

6.4 Commutative Banach algebra

7 Holomorphic functional calculus

8 C^* algebras

A *-algebra is a (complex) algebra A with an **involution**, i.e. a map $* : A \to A$ such that

- $(\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$,
- $(xy)^* = y^* x^*$,
- $x * * = x$, for every $x, y \in A$ and λ, μ are scalars.

Note that if A is unital, then $1^* = 1$.

Definition 8.1. A C^* algebra is a Banach algebra with an involution such that $\|xx^*\| = \|x\|^2$, for every $x \in A$.

Note that if A is unital, then $\|1\| = 1$.

Example 8.1. $C(K)$ is a C^* algebra with involution $f^*(x) = \overline{f(x)}$, where K is compact and Hausdorff.

Example 8.2. $\mathcal{B}(\mathcal{H})$ is a C^* algebra with involution $T* =$ adjoint operator of T, where $\mathcal{H}$ is a Hilbert space.

Example 8.3. A closed, *-subalgebra $\mathcal{B}$ of a C^* algebra is a C^* algebra. So all the closed *-subalgebra (C^* subalgebra) of $\mathcal{B}(\mathcal{H})$, are C^* subalgebras.