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MATH 600

GEOMETRIC ANALYSIS I

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Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

1 Manifolds

2 Integral curves and flows

2.1 Integral curves

Let M be a manifold,and X is a smooth vector field on M. An **integral curve** of X is a differentiable curve $\gamma : (-\epsilon, \epsilon) \to M$ such that

$$\gamma'(t) = X(\gamma(t)) \in T_{\gamma(t)}M \tag{2.1}$$

for each t in the domain.
Let $p \in M$, and (U, ϕ) be a chart containing p, so $\phi(p) = (x_1, \cdots, x_n)$. Let Y be a C^∞ vector field on M. Let $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$ be an integral curve, $\gamma(t) = (\gamma_1(t), \cdots, \gamma_n(t))$. We can write $Y(p) = (Y_1(p), \cdots, Y_n(p)$. By using $\gamma'(t) = X(\gamma(t))$ we get the following system of ODEs:

$$\begin{cases} \gamma_1'(t) = Y_1 \circ (\gamma_1(t), \cdots, \gamma_n(t)) \\ \gamma_2'(t) = Y_2 \circ (\gamma_1(t), \cdots, \gamma_n(t)) \\ \cdots \\ \gamma_n'(t) = Y_n \circ (\gamma_1(t), \cdots, \gamma_n(t)) \end{cases} \tag{*}$$

which is a system of n first order ODEs (usually nonlinear).

Theorem 2.1. *Let X be a smooth vector field on a manifold M. For any $p \in M$ there exists $a(p), b(p) \in \mathbb{R} \cup \{\pm\infty\}$ and a smooth curve*

$$\gamma_p : (a(p), b(p)) \to M \tag{2.2}$$

such that

- *$o \in (a(p), b(p))$, and $\gamma_p(0) = p$.*
- *γ_p is an integral curve of X.*
- *If $\mu : (c, d) \to M$ is a smooth curve satisfying the previous two conditions, then $(c, d) \subset (a(p), b(p))$. Moreover, $\mu = \gamma_p|_{(c,d)}$.*

Corollary 2.2. *Let X be a vector field on M. For any $p \in M$, there exists a neighborhood V of p and $a > 0$ such that $\Phi : (-a, a) \times V \to M$ is a C^∞ map, and $\Phi(t, p) = \gamma_p(t)$.*

Definition 2.1. For an integral curve $\gamma_p(a,b) \to M$, we define the **flow** of γ

$$\Phi_t(p) = \gamma_p(t). \tag{2.3}$$

Example 2.1. Let $Z(a) = a^2$ be a vector field on $\mathbb{R}$. Then we have $\gamma' = Z \circ \gamma = \gamma^2$. Solving this ODE gives $\gamma(t) = 1/(-t+c)$, so $\gamma(0) = 1/c = p$. The flow is given by

$$\Phi_t(p) = \frac{1}{-t + 1/p}. \tag{2.4}$$

Example 2.2. Let $Z(a) = a^{2/3}$. Now we don't have uniqueness. For example, take $\gamma(0) = 0$, then we have two solutions $\gamma(t) = 0$ and $\gamma(t) = (1/27)t^3$.

Corollary 2.3. *Let X be a vector field on M. For any $p \in M$, there exists a neighborhood V of p and $a > 0$ such that $\Phi : (-a,a) \times V \to M$ is a C^∞ map, and $\Phi(t,p) = \gamma_p(t)$.*

Proposition 2.4 (Group law)**.** *Let V and $a > 0$ be as in above, and $|t|$, $|s|$, $|t+s| < a$, then we have*

$$\Phi_s \circ \Phi_t = \Phi_{s+t} \tag{2.5}$$

at q if $q \in V$ and $\Phi_t(q) \in V$.

Proof. Fixed t. Define

$$\beta(s) = \Phi_{s+t}(q) \tag{2.6}$$
$$\delta_(s) = \Phi_s \circ \Phi_t(q). \tag{2.7}$$

We will show that both of them are integral curves with the same initial conditions. Put $s = 0$,

$$\beta(0) = \Phi_t(q) \tag{2.8}$$
$$\delta(0) = \Phi_0 \circ \Phi_t(q) = \Phi_t(q). \tag{2.9}$$

Note that the second equality follows from $\Phi_0(p) = \gamma_p(0) = p$.

$$\beta'(s_0) = \frac{d}{ds}\bigg|_{s=s_0} \beta(s) \tag{2.10}$$
$$= \frac{d}{ds}\bigg|_{s=s_0} \Phi_{s+t}(q) \tag{2.11}$$
$$= X\big(\Phi_{s_0+t}(q)\big) = X\big(\gamma_{\Phi_t(q)}(s_0)\big). \tag{2.12}$$

And

$$\delta'(s) = \frac{d}{ds}\bigg|_{s=s_0} \delta(s) \tag{2.13}$$
$$= \frac{d}{ds}\bigg|_{s=s_0} \Phi_s \circ \Phi_t(q) \tag{2.14}$$
$$= X\Big(\Phi_{s_0}\big(\Phi_t(q)\big)\Big) = X\big(\gamma_{\Phi_t(q)(s_0)}\big). \tag{2.15}$$

□

Definition 2.2. A vector field is **complete** if all its integral curves are defined for all $t \in \mathbb{R}$.

Theorem 2.5. *Let M be a manifold and X be a smooth vector field on M. If M is compact, or X has compact support, then X is complete.*

Proof. Suppose X has compact support, then there exists an open cover $\{U_i\}(i = 1, \cdots, k)$ of the support of X, with $a_i > 0$ such that $\Phi_i : (-a_i, a_i) \to M$ is the flow on each U_i. Let $a = \min_i\{a_i\}$. We have that each $\Phi_i : (-a, a) \to M$ is well defined. Φ_i and Φ_j agree on $U_i \cap U_j$ by the uniqueness of integral curves. Since X vanishes , every integral curve starting outside the support is constant and thus can be defined on all of t. Hence X is complete. □

If a vector field X is complete, then $\Phi_t : M \to M$ is a diffeomorphism. We have

$$\Phi_0 = Id \tag{2.16}$$
$$\Phi_t \circ \Phi_s = \Phi_{t+s} \tag{2.17}$$
$$(\Phi_t)^{-1} = \Phi_{-t}. \tag{2.18}$$

Proposition 2.6. *An* **one parameter family of diffeomorphism** *on M is a smooth map $\Phi_t : M \to M$ such that $\Phi_t \circ \Phi_s = \Phi_{t+s}$ and $\Phi_0 = Id$. This defines a vector field on M with flow Φ_t.*

Proof. Let $q \in M$. Define $Y(q) = \frac{d}{dt}\big|_{t=0}\Phi_t(q)$. Then Φ_t is the flow of $\gamma_q(t) = \Phi_t(q)$, and $\gamma_q(t)' = Y \circ \gamma_q(t)$ since

$$\frac{d\gamma}{dt}\bigg|_{t=t_0} \gamma_q(t) = \frac{d\gamma}{dt}\bigg|_{t=t_0} \Phi_q(t) \tag{2.19}$$
$$= \frac{d\gamma}{ds}\bigg|_{s=0} \Phi_{t_0+s}(q) \tag{2.20}$$
$$= \frac{d\gamma}{ds}\bigg|_{s=0} \Phi_s \circ \Phi_{t_0}(q) \tag{2.21}$$
$$= Y(\Phi_{t_0}(q)) = Y\big(\gamma_q(t_0)\big). \tag{2.22}$$

□

2.2 Lie derivatives

Let $f : M^n \to N^n$ be a diffeomorphism, and X be a vector field on X. Then we can get a **push forward** vector field $f_*(X)$ on N by setting

$$f_*(X)\Big|_q = df\Big|_{f^{-1}(q)}\big(X_{f^{-1}(q)}\big) \in T_qN \tag{2.23}$$

for some $q \in N$.
If Y is a vector field on N, then define the **pull back**

$$f^*(Y)\Big|_p = d(f^{-1})\Big|_{f(p)}\big(Y_{f(p)}\big) \in T_pM \tag{2.24}$$

for $p \in M$.

Exercise 2.1. Let f, X, Y be as above, and $g \in C^\infty(N)$. Show that

$$f_*(X)g = X(g \circ f) \circ f^{-1}. \tag{2.25}$$

Similarly, let $h \in C^\infty(M)$. We have

$$f^*(Y)h = X(h \circ f^{-1}) \circ f. \tag{2.26}$$

Lemma 2.7. *If X is a smooth vector field on M with $X_p \neq 0$, then there exists a coordinate (U, x) at p such that $X_q = \frac{\partial}{\partial x_1}\Big|_q$ for all $q \in U$.*

Proof. Choose a coordinate (V, y) containing p such that $y_1(p) = 0$ and

$$\frac{\partial}{\partial y_1}\Big|_p = X_p. \tag{2.27}$$

Let Φ_t be the flow of X. Define $\Psi : W \subset \mathbb{R}^n \to M$ by

$$\Psi(t, \cdots, a_n) = \Phi_t\big(y^{-1}(0, a_2, \cdots, a_n)\big). \tag{2.28}$$

Then we have

$$d\Psi\Big(\frac{\partial}{\partial r_1}\Big|_0\Big) f = \frac{d}{dt}\Big|_{t=0}\Big(f \circ \Phi_0\big(y^{-1}(0, a_2, \cdots, a_n)\big)\Big) \tag{2.29}$$

$$= X_{\Phi_t(y^{-1}\left(0, a_2, \cdots, a_n)\right)}\Big|_{t=0}(f) \tag{2.30}$$

$$= \frac{\partial}{\partial y_1}\Big|_p(f) \tag{2.31}$$

and

$$d\Psi\Big(\frac{\partial}{\partial r_i}\Big|_0\Big) = \frac{\partial}{\partial y_i}\Big|_p \tag{2.32}$$

for $f \in C^\infty(M)$. Note that $\Psi(0, \cdots, a_n) = y^{-1}\big((0, \cdots, a_n)\big)$. Hence $d\Psi\big|_{t=0}$ is invertible hence an isomorphism. By Inverse function theorem, there exists an open $U \subset \Psi(W)$ such that $\Psi : U \to \Psi(U)$ is a diffeomorphism. Therefore $x = \Psi^{-1}$ is a coordinate function. Let $q \in U$ and $x(q) = (a_1, \cdots, a_n)$.

$$X_q = X\Big(\Psi\big((a_1, \cdots, a_n)\big)\Big) = \frac{d}{dt}\bigg|_{t=0} \Phi_t\Big(\Psi\big((a_1, \cdots, a_n)\big)\Big) \tag{2.33}$$

$$= \frac{d}{dt}\bigg|_{t=0} \Phi_t \circ \Phi_{a_1}\big(y^{-1}(0, a_2, \cdots, a_n)\big) \tag{2.34}$$

$$= \frac{d}{dt}\bigg|_{t=0} \Phi_{t+a_1}\big(y^{-1}(0, a_2, \cdots, a_n)\big) \tag{2.35}$$

$$= \frac{d}{dt}\bigg|_{t=0} \Psi\big((a_1 + t, a_2, \cdots, a_n)\big) \tag{2.36}$$

$$= \frac{\partial}{\partial x_1}\bigg|_q \tag{2.37}$$

as required. □

Definition 2.3. Let X, Y be vector fields on M, and X has flow Φ_t. Then we define $L_X Y$ to be the **Lie derivative** of Y with respective to X at p which has the expression

$$(L_X Y)_p = \lim_{p \to 0} \frac{\Phi_t^*(Y_{\Phi_t(p)}) - Y_p}{t} = \frac{d}{dt}\bigg|_{t=0} \big(\Phi_t^*(Y)_{\Phi_t(p)}\big). \tag{2.38}$$

Exercise 2.2. Show that $L_X(f) = X(f)$.

Theorem 2.8. *Let X be a vector field on M with flow Φ_t, and Y be another vector field on X. Let $p \in M$, then*

$$[X, Y](p) = \frac{d}{dt}\bigg|_{t=0} \big(\Phi_t^*(Y_{\Phi_t(p)})\big) = \frac{d}{dt}\bigg|_{t=0} \Big(d(\Phi_{-t})\big(Y \circ \Phi_t(p)\big)\Big). \tag{2.39}$$

Proof. WLOG, assume that $X_p \neq 0$. Choose coordinate (U, z) containing p such that $X = \frac{\partial}{\partial z_1}$ on U. Let Y be a vector field on X, so $Y = \sum_i b_i \frac{\partial}{\partial z_i}$ on U. Let Φ_t be the flow of X on U. Then

$$\Phi_t \circ z^{-1}(a_1, \cdots, z_n) = z^{-1}(a_1 + t, a_2, \cdots, z_n) \tag{2.40}$$

and

$$d(\Phi_{-t})\Big(\frac{\partial}{\partial z_i}\bigg|_q\Big) = \frac{\partial}{\partial z_i}\bigg|_{\Phi_{-t}(q)} \tag{2.41}$$

for some $q \in U$ and $z(q) = (a_1, \cdots, z_n)$. Hence,

$$d(\Phi_{-t})\Big(Y\Big|_{\Phi_t(p)}\Big) = \sum_{i=1}^n b_i\big(z^{-1}(t, 0, \cdots, 0)\big)\frac{\partial}{\partial z_i}\bigg|_p. \tag{2.42}$$

Now

$$\frac{d}{dt}\Big|_{t=0} d(\Phi_{-t})\big(Y \circ \Phi_t(p)\big)\Big) = \sum_{i=1}^{n} \frac{d}{dt}\Big|_{t=0}\Big(b_i\big(z^{-1}(t,0,\cdots,0)\big)\frac{\partial}{\partial z_i}\Big|_p\Big) \tag{2.43}$$

$$= \sum_{i=1}^{n} \frac{\partial b_i}{\partial z_1}\frac{\partial}{\partial z_i}\Big|_p = [X,Y]\Big|_p. \tag{2.44}$$

□

Remark 2.1. Let $f : M \to N$ be a diffeomorphism, and Y is a vector field on N. Then $f^*(Y)$ is a vector field on M. Let $p \in M$, we have $f^*(Y)\big|_p = df^{-1}Y_{f(p)}$. Note that $df\big(f^*(Y)\big) = Y$.

Exercise 2.3. If Ψ_t is the flow of Y on N and $f : M \to N$ is a diffeomorphism, then the flow of $f^*(Y)$ is $f^{-1} \circ \Psi_t \circ f$.
Similarly, if Φ_t is the flow of X on M, then the flow of $f_*(X)$ is $f \circ \Phi_t \circ f^{-1}$.

Theorem 2.9. *If Φ_t is a flow of X and Ψ_t is a flow of Y, then $[X,Y] = 0$ if and only if*

$$\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t \tag{2.45}$$

for all t, s. Equivalently, we have

$$\Phi_t \circ \Psi_s \circ \Phi_{-t} \circ \Psi_{-s} = Id. \tag{2.46}$$

Corollary 2.10. *Let $c(t) = \Psi_{-t}\circ\Phi_{-t}\circ\Psi_t\circ\Phi_t(q)$, then $c(0) = q$, $c'(0) =)$, and $c''(0) = \frac{1}{2}[X,Y]$.*

Theorem 2.11. *Given vector fields $X_1, \cdots, X_n$ on M which are linearly independent for every $p \in M$. Then there exists a coordinate (U, x) containing p such that $X(q) = \frac{\partial}{\partial x_i}\Big|_q$ for all $q \in U$ if any only if $[X_i, X_j] = 0$.*

2.3 Distributions

3 Lie groups and Lie algebras

3.1 Lie groups

Definition 3.1. A **Lie group** is a smooth manifold which also has the group structure and the map $G \times G \to G$ defined by $(a,b) \to ab^{-1}$ is smooth for $a, b \in G$.

Example 3.1. $(\mathbb{R}^n, +)$ is clearly a Lie group.

Example 3.2. $GL(n) = \{A \in M_n(\mathbb{R}) : \det A \neq 0\}$ as an open subset of $\mathbb{R}^{n^2}$ is a Lie group.

Example 3.3. $SL(n,\mathbb{R})$, $O(n)$, $SO(n)$ are all Lie groups (are embedded submanifolds of $\mathbb{R}^{n^2}$.)

Theorem 3.1. *If G is a Lie group, and $H \subset G$ is a subgroup, then H is a Lie group.*

Corollary 3.2. *Any subgroups of $GL(n,\mathbb{R})$ defined by equations are Lie groups. For example,*

$$O(n) = \{A \in GL(n,\mathbb{R}) : AA^T = I\}, \tag{3.1}$$
$$U(n) = \{A \in GL(n,\mathbb{C}) : A\bar{A}^T = I\}, \tag{3.2}$$
$$SU(n) = \{A \in U(n) : \det A = 1\}. \tag{3.3}$$

We call the subgroups of $GL(n,\mathbb{R})$ by **matrix groups**.

Definition 3.2. The **quaternion** $\mathbb{H}$ is a four dimensional vector space defined by

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\} \tag{3.4}$$

with relations $i^2 = j^2 = k^2 = ijk = -1$. Since $-1 = ijk$, we have

$$-k = ijkk = ij(k^2) = ij(-1), \tag{3.5}$$
$$k = ij. \tag{3.6}$$

All the other possible products can be determined by similar methods, resulting in

$$ij = k, ji = -k, \tag{3.7}$$
$$jk = i, kj = -i, \tag{3.8}$$
$$ki = j, ik = -j. \tag{3.9}$$

The conjugate of $q = a + bi + cj + dk \in \mathbb{H}$ is $\bar{q} = a - bi - cj - dk$. By calculation we have $q\bar{q} = a^2 + b^2 + c^2 + d^2$. Moreover, if $q_1, q_2 \in \mathbb{H}$, then $|q_1 \cdot q_2| = |q_1| \cdot |q_2|$.

Let $GL(n,\mathbb{H})$ be $n \times n$ matrices over $\mathbb{H}$ which have inverses. In this case, $\det AB$ may not equals to $\det A \det B$ due to non-commutitivity of $\mathbb{H}$. We define $Sp(n) = \{A \in GL(n,\mathbb{H}) : A\bar{A}^T = I\}$ to be a subgroup of $GL(n,\mathbb{H})$.
If $A \in U(n)$, then there exists $B \in U(n)$ such that

$$BAB^{-1} = diag(z_1, \cdots, z_n), \ |z_k| = 1. \tag{3.10}$$

If $A \in O(n)$, A may not be diagonalizable. But there exists some $B \in O(n)$ such that we can write

$$BAB^{-1} = \left(\begin{array}{c|c|cc} R^{\theta_1} & 0 & 0 & \cdots \\ \hline 0 & R^{\theta_2} & 0 & \cdots \\ \hline 0 & a_{i2} & \ddots & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}\right), \tag{3.11}$$

where R^{θ_i} is either ± 1 or the usual rotation

$$R^{\theta_i} = \begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix}. \tag{3.12}$$

Exercise 3.1. For $A \in Sp(n)$, does there exist $B \in Sp(n)$ such that $BAB^{-1} = diag(a_1, \cdots, a_n)$ for $q_i \in \mathbb{H}$ and $|q_i| = 1$?

Theorem 3.3 (Hilbert's 5th problem)**.** *Suppose G is a topological group, i.e. G is both a group and topological manifold, and the map $G \times G \to G$ defined by $(a, b) \to ab^{-1}$ is continuous, then G is a Lie group.*

Theorem 3.4. *Assume G is a C^2 Lie group, then G has a unique analytic structure.*

3.2 Lie algebras

Let G be a Lie group and $g \in G$. The **left translation** by g is a map $L_g : G \to G$ defined by $L_g(h) = gh$, for any $h \in G$. L_g is in fact a diffeomorphism on G, and we have $(L_g)^{-1} = L_{g^{-1}}$, $L_g \circ L_h = L_{gh}$. Similarly, we can define the **right translation** $R_g : G \to G$ by $R_g(h) = hg$. Note that $L_g \circ R_{g^{-1}}$ is the conjugate map.

Definition 3.3. Let X be a vector field on a Lie group G. X is called **left invariant** if it is smooth and $L_{g_*}X = X$, that is,

$$(dL_g)_h\big(X(h)\big) = X(gh) \tag{3.13}$$

for any $g, h \in G$.

If X and Y are left invariant, then $[X, Y]$ is also left invariant. Since X and $L_{g_*}X$ are L_g-related,

$$L_{g_*}\big([X, Y]\big) = [L_{g_*}(X), L_{g_*}(Y)]. \tag{3.14}$$

Definition 3.4. Let V be a vector space, then $\mathfrak{g} = (V, [\ ,\ ])$ is called a **Lie algebra** , if the **bracket** $[\ ,\ ] : V \times V \to V$ satisfies

1. $[v, w] = -[w, v]$. (anti-commutativity)
2. $[\ ,\ ]$ is bilinear.
3. $[[v, w], z] + [[w, z], v] + [[z, v], w] = 0$. (Jabobi identity)

for all $v, w, z \in V$.

Example 3.4. The euclidean space $\mathbb{R}^3$ with the cross product as bracket is a Lie algebra.

Example 3.5. $M_n(\mathbb{R})$ is a Lie algebra with the bracket

$$[A, B] = AB - BA \tag{3.15}$$

for $A, B \in M_n(\mathbb{R})$.

Example 3.6. Let X be the collection of C^∞ vector fields on M, then X is an infinite dimensional Lie algebra.

Theorem 3.5 (Ado)**.** *If $\mathfrak{g} = (V, [\ ,\ ])$ is a finite dimensional Lie algebra, then there exists a subalgebra of $M_n(\mathbb{R})$ that is isomorphic to $\mathfrak{g}$.*

If G is a Lie group, then the set of left invariant vector fields form a Lie algebra.

Theorem 3.6. *Let G be a Lie group. For each $v \in T_eG$, there exists a unique left invariant vector field X such that $X(e) = v$. Hence the dimension of left invariant vector fields on G equals the dimension of G. In fact, $X \to X(e)$ is an isomorphism.*

Proof. Let $v \in T_eG$. Define $X(g) = (dL_g)_e(v)$. We need to show that X is left invariant and smooth. Since

$$(dL_g)_h\big(X(h)\big) = (dL_g)_h\big((dL_h)_e(v)\big) \tag{3.16}$$

$$= (dL_gh)_e(v) = X(gh), \tag{3.17}$$

Hence X is left invariant. Then we need show X is a smooth vector field. It suffices to show Xf is smooth whenever $f \in C^\infty(G)$. Let $\gamma : (-\epsilon, \epsilon) \to G$ be a smooth curve such that $\gamma(0) = e$, $\gamma'(0) = v$. Then we have

$$Xf(g) = (dL_g)_e(v)f = v(f \circ L_g) \tag{3.18}$$

$$= \gamma'(0)(f \circ L_g) \tag{3.19}$$

$$= \frac{d}{dt}\Big|_{t=0}\big(f \circ L_g \circ \gamma(t)\big). \tag{3.20}$$

We can define $\phi : (-\epsilon, \epsilon) \times G \to \mathbb{R}$ by

$$\phi(t, g) = f \circ L_g \circ \gamma(t). \tag{3.21}$$

Then we see that $Xf(g) = \frac{\partial \phi}{\partial t}(0, g)$. Since f, γ, and the left multiplication L_g are all smooth, it follows that $\frac{\partial \phi}{\partial t}(0, g)$ is smooth.
The injectivity of $X \to v \in \mathfrak{g}$ is clear. Hence the set of left invariant vector fields on G is isomorphic to $\mathfrak{g} = T_eG$. □

Proposition 3.7. *Let $\mathfrak{gl}(n, \mathbb{R}) = T_e\big(GL(n, \mathbb{R})\big)$, then $\mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R})$ with bracket $[A, B] = AB - BA$ for $A, B \in \mathfrak{gl}(n, \mathbb{R})$.*

Proof. Let $X \in T_e\big(GL(n, \mathbb{R})\big) \simeq M_n(\mathbb{R})$, and $\tilde{X}$ be the induced left invariant vector field of X on $GL(n, \mathbb{R})$. Let $A \in GL(n, \mathbb{R})$. Then

$$\tilde{X}(A) = (dL_A)_eX = A \cdot X. \tag{3.22}$$

Let x_{ij} be coordinate functions in $M_n(\mathbb{R})$, then $x_{ij}(A) = A_{ij}$. Since $\tilde{X}$ is a vector field, we can write it as

$$\tilde{X} = \sum \tilde{X}(x_{ij})\frac{\partial}{\partial x_{ij}}. \tag{3.23}$$

Note that

$$\tilde{X}\big(x_{ij}(A)\big) = (dx_{ij})_A\big(\tilde{X}(A)\big) \tag{3.24}$$

$$= x_{ij}\big(\tilde{X}(A)\big) = (AX)_{ij}. \tag{3.25}$$

Hence we have that $\tilde{X}(e) = X$, $\tilde{X}(A) = \sum (AX)_{ij}\frac{\partial}{\partial x_{ij}}$. Let $Y \in T_e\big(GL(n, \mathbb{R})\big)$ and $\tilde{Y}$ be the corresponding induced left invariant vector field. Then

$$[\tilde{X},\tilde{Y}]_e = (\tilde{X}\tilde{Y} - \tilde{Y}\tilde{X})_e \tag{3.26}$$
$$= \tilde{X}_e(A \to AY) - \tilde{Y}_e(A \to AX). \tag{3.27}$$

Using the coordinate function

$$[\tilde{X},\tilde{Y}]_e(x_{ij}) = \tilde{X}_e(A \to AY)_{ij} - \tilde{Y}_e(A \to AX)_{ij} \tag{3.28}$$
$$= (XY - YX)_{ij}. \tag{3.29}$$

Hence $[\tilde{X},\tilde{Y}]_e = XY - YX$. The result then follows from the previous theorem. □

Theorem 3.8. *Let $G \subset GL(n,\mathbb{R})$ be a closed subgroup (hence a Lie group). Then $\mathfrak{g} = T_eG$ has bracket $[A,B] = AB - BA$ for $A, B \in \mathfrak{g}$.*

Example 3.7. Consider $SL(n,\mathbb{R}) \subset GL(n,\mathbb{R})$. We have

$$\mathfrak{sl}(n,\mathbb{R}) = T_e\big(SL(n,\mathbb{R})\big) \subset T_e\big(GL(n,\mathbb{R})\big) = \mathfrak{gl}(n,\mathbb{R}). \tag{3.30}$$

In fact,

$$\mathfrak{sl}(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) : tr(A) = 0\}. \tag{3.31}$$

In order to prove the above proposition, we take a path $A(t) \in SL(n,\mathbb{R})$ with $A(0) = I$, $A'(0) \in \mathfrak{sl}(n,\mathbb{R})$. Fact: Since $D(\det)_A(B) = \det(A) \cdot tr(B)$, we have $D(\det)_I(B) = tr(B)$. Using this fact,

$$\frac{d}{dt}\Big|_{t=0} \det A(t) = 0 = D(\det)_I\big(A'(0)\big) \tag{3.32}$$
$$= tr\big(A'(0)\big). \tag{3.33}$$

Therefore $T_e\big(SL(n,\mathbb{R})\big) \subset \{A \in M_n(\mathbb{R}) : trA = 0\}$. The equality follows by counting the dimension. More generally, we can prove the above result by using the exponential map.

Definition 3.5. Let A be a complex $n \times n$ matrix, we define the **exponential map** of A by

$$e^A = \exp(A) = I + A + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \cdots \tag{3.34}$$

It is easy to show that the above series converges absolutely and uniformly on any compact set $|A| < \alpha$ for any $\alpha > 0$.

Proposition 3.9. *The exponential map has the following properties:*

1. $\det e^A = e^{tr(A)}$.
2. $e^A \cdot e^B = e^{A+B} = e^B \cdot e^A$ *if any only if* $AB = BA$.
3. $Be^AB^{-1} = e^{BAB^{-1}}$.
4. $Be^AB^T = e^{BAB^T}$.

Note that in $M_n\mathbb{R}$, the matrices that can be diagonalized are dense.

Example 3.8. Let $\mathfrak{o}(n)$ be the Lie algebra of $O(n)$, then $\mathfrak{o}(n) = \{A : A + A^T = 0\}$. Note that $SO(n)$ is a component of $O(n)$ that contains the identity. Let $A(t)$ be a path in $SO(n)$ such that $A(0) = I$ and $A'(0) \in \mathfrak{o}(n)$, then $A(t)A^T(t) = I$. Differentiate both sides at 0 gives

$$\Big(A'(t)A(t) + A(t)\big(A^T(t)\big)'\Big)\Big|_{t=0} = 0, \tag{3.35}$$

since $A(0) = I$,

$$A'(0) + \big(A^T(0)\big)' = 0. \tag{3.36}$$

This shows that if $A \in \mathfrak{o}(n)$, then $A + A^T = 0$. For the other direction, let $A \in \mathfrak{o}(n)$, and define the map $t \mapsto e^{tA} = B(t)$. From the definition of the exponential map,

$$e^{tA} == I + tA + \frac{t^2A^2}{2!} + \cdots + \frac{t^nA^n}{n!} + \cdots \tag{3.37}$$

we have $B(0) = I$ and $B'(0) = A$. Then

$$B(t)B^T(t) = e^{tA}e^{tA^T} = e^{tA+tA^T} = e^0 = I \tag{3.38}$$

since A and A^T commute. We are done.

Theorem 3.10. *If $G \subset GL(n, \mathbb{R})$, so $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$. Then $A \in \mathfrak{g}$ if and only of $e^A \in G$.*

3.3 Homomorphisms

Definition 3.6. A map $\phi : H \to G$ is **Lie group homomorphism** if it is a group homomorphism and it is smooth.

Definition 3.7. $H \subset G$ is a **Lie subgroup** if the inclusion map $H \hookrightarrow G$ is both a Lie group homomorphism and a 1-1 immersion, i.e. a 1-1 immersed submanifold + subgroup.

Theorem 3.11. *Let H, G be Lie groups with Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$ respectively, and let $\phi : H \to G$ be a homomorphism. Then X and $d\phi(X)$ are ϕ-related, and $(d\phi)_e : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism, i.e.*

$$(d\phi)_e([X, Y]) = [(d\phi)_e(X), (d\phi)_e(Y)] \tag{3.39}$$

for $X, Y \in \mathfrak{h}$.

Proof. Let $X, Y \in \mathfrak{h} = T_eH$. Define $\tilde{X}, \tilde{Y}$ be the corresponding left invariant vector fields with $\widetilde{X(e)} = X$ and $\tilde{Y}(e) = Y$. Note that $(dL_g)_eX = \tilde{X}(g)$ for $g \in H$. First we show that $\tilde{X}$ and $\widetilde{(d\phi)_e}X$ are ϕ - related, i.e.

$$(d\phi)_g\big(\tilde{X}(g)\big) = \widetilde{(d\phi)_e}X\big(\phi(g)\big) \tag{3.40}$$

for all $g \in H$.
By definition,

$$(d\phi)_g\big(\tilde{X}(g)\big) = (d\phi)_g\big((dL_g)_e(X)\big) \tag{3.41}$$
$$= d(\phi \circ L_g)_e(X). \tag{3.42}$$

Since

$$\phi\big(L_g(h)\big) = \phi(gh) = \phi(g)\phi(h) = L_{\phi(g)}\big(\phi(h)\big), \tag{3.43}$$

we have $\phi \circ L_g = L_{\phi(g)} \circ \phi$. Therefore,

$$d(\phi \circ L_g)_e(X) = d\big(L_{\phi(g)} \circ \phi\big)_e(X) \tag{3.44}$$
$$= (dL_{\phi(g)})(d\phi)_e(X) \tag{3.45}$$
$$= \widetilde{(d\phi)_e X} \circ \phi(g), \tag{3.46}$$

since $(d\phi)_e(X) \in \mathfrak{g}$. Hence we have proved that $\tilde{X}$ and $\widetilde{(d\phi)_e X}$ are ϕ - related, which implies that X and $d\phi(X)$ are ϕ -related.
Now we need show

$$(d\phi)_e([\tilde{X}, \tilde{Y}]_e) = [\widetilde{(d\phi)_e(X)}, \widetilde{(d\phi)_e(Y)}]_e. \tag{3.47}$$

This follow directly from X, Y and $d\phi(X)$, $d\phi(Y)$ are ϕ -related respectively. □

Corollary 3.12. *If $H \subset G$ is a Lie subgroup, then $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra.*

Example 3.9. Let $G \subset GL(n, \mathbb{R})$ be a Lie subgroup (i.e. a closed subgroup), then for $A, B \in \mathfrak{g}$, $[A, B] = AB - BA$. Moreover, $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$.

Example 3.10. The unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ a Lie group.

Example 3.11. $T^2 = S^1 \times S^1$ is also a Lie group. We can also view $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. If H has irrational slope, then it is a 1-1 immersed submanifold.

Theorem 3.13. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Then there exists a unique connected Lie subgroup $H \subset G$ such that the Lie algebra of H is $\mathfrak{h}$.*

Proof. Let Δ be the distribution of G such that $\Delta_e = h \in \mathfrak{h}$, and Δ is left invariant, i.e.

$$\Delta_g = (dL_g)_e(h). \tag{3.48}$$

Δ is a smooth distribution since it is globally spanned by left invariant vector fields $\tilde{X}_i$ on G such that $\tilde{X}_i(e) \in \mathfrak{h} = \Delta_e$ and $\tilde{X}(g) \in \Delta_g$. Since $[\tilde{X}, \tilde{Y}]$ is left invariant and $[\tilde{X}, \tilde{Y}] = [X, Y] \in \mathfrak{h}$ which follows from the fact that $\mathfrak{h}$ is a subalgebra, i.e. $X, Y \in \mathfrak{h} \Rightarrow [X, Y]_{\mathfrak{g}} \in \mathfrak{h}$, Δ is involutive hence integrable.
Let H be a maximal leaf of Δ through $e \in G$, then this leaf is unique. We will show H is a subgroup of G. Let $h \in H$, then $h^{-1} \in G$. We claim that $L_{h^{-1}}(H) = H$.

Lemma 3.14. *For any F being a leaf of Δ in G, then $L_g(F)$ is also a leaf, where $g \in G$.*

Proof of the lemma. Since $F \subset G$, $T(F)_g = \Delta_g$, so

$$T\big(L_g(F)\big)_{\bar{g}} = d(L_g)\big(T(F)_{\bar{g}}\big) = d(L_g)\Delta_{\bar{g}} = \Delta_{g\bar{g}}. \tag{3.49}$$

By this lemma L_g sends leaves of Δ into leaves of Δ. So $L_{h^{-1}}(H)$ is a leaf of Δ. Since $L_{h^{-1}}(h) = e$, $L_{h^{-1}}(H)$ is a leaf through $e \in G$. By uniqueness of the maximality, $L_{h^{-1}}(H) = H$. Therefore, if $h, g \in H$, we have $h^{-1}g \in H$. We see that H is an abstract subgroup of G, hence a Lie subgroup. □

Corollary 3.15. *Let $\phi, \psi : H \to G$ be two Lie group homomorphisms with $(d\phi)_e = (d\psi)_e$, then $\phi = \psi$.*

Proof. Let $graph(\phi) = \{\big(h, \phi(h)\big) \in H \times G : h \in H\}$. Note that $H \times G$ is a Lie group, ϕ is a homomorphism iff $graph(\phi)$ is a subgroup of $H \times G$, since

$$\big(h, \phi(h)\big) \cdot \big(\tilde{h}, \phi(\tilde{h})\big) = \big(h\tilde{h}, \phi(h)\phi(\tilde{h})\big) = \big(h\tilde{h}, \phi(h\tilde{h})\big). \tag{3.50}$$

Also, $l : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism iff $graph(l) = \{\big(h, l(h)\big) : h \in \mathfrak{h}\}$ is a Lie subalgebra of $\mathfrak{h} \times \mathfrak{g}$.
Now $graph(\phi)$, $graph(\psi)$ are Lie subgroups of $H \times G$, and $graph(d\phi)$ is a Lie subalgebra of $\mathfrak{g}$. We have

$$T_e\big(graph(\phi)\big) = graph\big((d\phi)\big)_e, \tag{3.51}$$

since if we let $\Big(g(t), \phi\big(g(t)\big)\Big))$ and $g(0) = e$, then $graph\big(g'(0)\big) = \Big(g'(0), \phi\big(g'(0)\big)\Big)$.
Now $graph(\phi)$ and $graph(\psi)$ are two Lie subgroups of $H \times G$ and have the same Lie algebra. By uniqueness, $\phi = \psi$.

□

Question 1 *Given a Lie algebra $\mathfrak{g}$, does there exist a Lie group G which has the Lie algebra $\mathfrak{g}$?* This question is answered by Ado's theorem.

Question 2 *Given two Lie groups H and G, let $\phi : \mathfrak{h} \to \mathfrak{g}$ be a homomorphism of the Lie algebras of H and G respective. Does there exist a homomorphism $\psi : H \to G$ with $d\psi = \phi$?*

Question 3 *Given $\mathfrak{g}$, is G unique?* The answer is no. For example, $SU(2)$, $SO(3)$, and $Sp(1)$ have the same Lie algebra. We have $SU(2) \simeq Sp(1)$, whereas $Sp(1) \mapsto SO(3)$ defined by $q \mapsto \{v \to qv\bar{q}\}$ is a 2-1 map.

Exercise 3.2. Let $q \in Sp(1)$, and write $q = a + bj$ for $a, b \in \mathbb{C}$. Show that

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in SU(2).$$

4 Differential forms

5 Integration on manifolds

5.1 Integration

Let $f : \mathbb{R}^n \to \mathbb{R}^n$, and $U \subset \mathbb{R}^n$, then

$$\int_U f \ dx_1 \cdots dx_n = \int \cdots \int (f \ dx_1) dx_2 \cdots dx_n. \tag{5.1}$$

Let $g : U \to V$ be a local diffeomorphism. Let $U, V \subset \mathbb{R}^n$ be open. Applying the change of variables formula, we get

$$\int_V f \ dx_1 \cdots dx_n = \int_{g(V)} f \circ g \ |\det Dg| dy_1 \cdots dy_n. \tag{5.2}$$

Now let M be an oriented n-dimensional manifold. Let $\omega \in \Omega^n(M)$, our aim is to define $\int_M \omega$.
Locally, we can write $\omega = f dx_1 \cdots dx_n$, where (U, x) is a local coordinate. Then we define

$$\int_U \omega = \int_{x(U)} f \circ x^{-1} \ dx_1 \cdots dx_n. \tag{5.3}$$

In another coordinate (V, y) for $U \cap V \neq \emptyset$. Suppose that *supp* $\omega \subset U \cap V$, then $\omega \in \Omega^n(U), \Omega^n(V)$. By previous definition,

$$\int_U \omega = \int_{x(U)} f \circ x^{-1} \ dx_1 \cdots dx_n; \tag{5.4}$$

$$\int_V \omega = \int_{y(V)} f \circ y^{-1} \ dy_1 \cdots dy_n. \tag{5.5}$$

Let $g = x \circ y^{-1} : y(U \cap V) \to x(U \cap V)$, then it is a diffeomorphism. The pull-back

$$g^*(dx_1 \wedge \cdots \wedge dx_n) = D(g)^*(dx_1 \wedge \cdots \wedge dx_n) \tag{5.6}$$

$$= \det(Dg)(dy_1 \wedge \cdots \wedge dy_n). \tag{5.7}$$

Lemma 5.1. *Let $x = g(y)$, Dg preserves the orientation, then*

$$\int_{g(U \cap V)} \omega = \int_{U \cap V} g^*(\omega). \tag{5.8}$$

Proof.

$$
\begin{aligned}
g^*(\omega) &= g^*(f dx_1 \wedge \cdots \wedge dx_n) && (5.9)\\
&= D(g)^*(f dx_1 \wedge \cdots \wedge dx_n) && (5.10)\\
&= \det(Dg)(f \circ x^{-1} dx_1 \wedge \cdots \wedge dx_n) && (5.11)\\
&= \det(Dg)(f \circ x^{-1} \circ x \circ y^{-1} dy_1 \wedge \cdots \wedge dy_n) && (5.12)\\
&= \det(Dg)(f \circ y^{-1} dy_1 \wedge \cdots \wedge dy_n). && (5.13)
\end{aligned}
$$

Then we just need to app ly the change of variables formula as before. □

Let M be an orientable manifold with oriented atlas (U_α, x_α). Choose a partition of unity $\{\phi_\alpha\}$ subordinate to $\{U_\alpha\}$, with $\phi_\alpha : M \to \mathbb{R}$, $\sum \phi_\alpha = 1$, and *supp* $\phi_\alpha \subset U_\alpha$. If $\omega \in \Omega^n(M)$ has compact support, we define

$$\int_M \omega = \int \Big(\sum_\alpha \phi_\alpha\Big)\omega = \sum_\alpha \Big(\int_{U_\alpha} \phi_\alpha \omega\Big), \tag{5.14}$$

where

$$\int_U \omega = \int_{x(U)} f \circ x^{-1}\ dx_1 \wedge \cdots \wedge dx_n \tag{5.15}$$

as before. We need to show that this formula is independent of the choice of the partition of unity and the coordinates. Suppose we have another partition of unity $\{\psi_\beta\}$ subordinate to charts $\{V_\beta\}$.

$$
\begin{aligned}
\int_M \omega &= \sum_\alpha \Big(\int_{U_\alpha} \phi_\alpha \omega\Big) && (5.16)\\
&= \sum_\alpha \Big(\int_{U_\alpha} \big(\sum_\beta \eta_\beta\big)\phi_\alpha \omega\Big) && (5.17)\\
&= \sum_{\alpha,\eta} \Big(\int_{U_\alpha \cap V_\beta} \eta_\beta \phi_\alpha \omega\Big) && (5.18)\\
&= \sum_{\eta,\alpha} \Big(\int_{U_\alpha \cap V_\beta} \phi_\alpha \eta_\beta \omega\Big) && (5.19)\\
&= \sum_\beta \Big(\int_{V_\beta} \big(\sum_\alpha \phi_\alpha\big)\eta_\beta \omega\Big) && (5.20)\\
&= \sum_\beta \Big(\int_{V_\beta} \eta_\beta \omega\Big). && (5.21)
\end{aligned}
$$

Theorem 5.2. *Let $F : M \to N$ be an orientation preserving diffeomorphism, $\omega \in \Omega^n(N)$. Then we have*

$$\int_M f^*(\omega) = \int_{f(M)=N} \omega. \tag{5.22}$$

Example 5.1 (line integrals in $\mathbb{R}^3$)**.** Let $c : I \to \mathbb{R}^3$ be a smooth parametrized curve, where $I \subset \mathbb{R}$ is an interval. Let $f : N^k \to M^n$, $k < n$ be a smooth map, and $\omega \in \Omega^n(M)$. We define

$$\int_{N^k} f^*(\omega) = \int_{f(N^k)} \omega. \tag{5.23}$$

Hence we could define the line integral of $\omega = f_1 dx + f_2 dy + f_3 dz$ over c in $\mathbb{R}^3$ by

$$\int_C f_1 dx + f_2 dy + f_3 dz = \int_I c^*(\omega). \tag{5.24}$$

Note that $c = \big(x(t), y(t), z(t)\big)$ is independent of parametrization, and $c' = (x', y', z')$. So we have

$$\int_I c^*(\omega) = \int_I f_1 \frac{dx}{dt} dt + f_2 \frac{dy}{dt} dt + f_3 \frac{dz}{dt} dt \tag{5.25}$$

$$= \int_I \Big(\sum_i f_i \frac{dx_i}{dt} \Big) dt. \tag{5.26}$$

Example 5.2 (surface integral in $\mathbb{R}^3$)**.** Let $f : U \subset R^3 \to \mathbb{R}$, and $x(s,t)$ be a smooth parametrization of $M^2 \in \mathbb{R}^3$. Then the surface integral of f is

$$\int_M^2 f dA = \int_{\mathbb{R}^2(s,t)} x^*(f dA), \tag{5.27}$$

where dA is the area element of M^2

$$dA = \|\frac{dx}{ds} \times \frac{dx}{dt}\| ds dt. \tag{5.28}$$

5.2 Stokes' theorem

Let $\mathbb{H}^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_n \geq 0)\}$ be the upper half plane. Then the boundary of $\mathbb{H}^n$ is $\partial\mathbb{H}^n = \{x_n = 0\} \simeq \mathbb{R}^{n-1}$. We say M is a **manifold with boundary** ∂M if for each $p \in \partial M$, there exist coordinates (U, x), (V, y) of p, such that $x : U \to \mathbb{H}^n$, $y : V \to \mathbb{H}^n$ are homeomorphisms, $x(p), y(p) \in \partial\mathbb{H}^n$, moreover, $x \circ y^{-1} : \mathbb{H}^n \to \mathbb{H}^n$ is a diffeomorphism in the interior of $\mathbb{H}^n$ and $x \circ y^{-1}(\partial\mathbb{H}^n) \subset \partial\mathbb{H}^n$.
An orientation on M induces an orientation on ∂M. In fact, let $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \in T_pM$, then $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{n-1}} \in T_p(\partial M)$ induces an orientation on ∂M. Conversely, let $v_1, \cdots, v_{n-1} \in T_p(\partial M) \subset T_pM$ be an oriented basis, and let $\vec{n}$ is the outer normal vector. Then $v_1, \cdots, v_{n-1}, \vec{n}$ is an oriented basis for T_pM.

Theorem 5.3 (Stokes')**.** *Let M be an orientable manifold with boundary ∂M. We have*

$$\int_M d\omega = \int_{\partial M} \omega. \tag{5.29}$$

Let $i : \partial M \to M$ be a smooth map which is an identity when restricted to ∂M, then we have $i^*(\omega) \in \Omega^{n-1}(\partial M)$, and

$$\int_M d\omega = \int_{\partial M} i^*(d\omega) = \int_{\partial M} \omega. \tag{5.30}$$

Theorem 5.4 (Brower fixed point theorem)**.** *Let $B^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 \leq 1\}$. Let $f : B^n \to B^n$ be a C^∞ map, then there exists a point $x \in B^n$ such that $f(x) = x$.*

Proof. Suppose f has no fixed points. Let $j : \partial B^n \to B^n$ such that $j\big|_{\partial B^n} = Id$, where $j(x)$ is defined to be the intersection of ∂B^n and the ray starting from $f(x)$ through x. If f is C^∞, then j is C^∞. We need to choose a $\omega \in \Omega^n(B)$ such that $\int_{\partial B^n} \omega > 0$, then we have

$$0 \neq \int_{\partial B^n} \omega = \int_{\partial B^n} j^*(\omega) \tag{5.31}$$

since $j\big|_{\partial B^n} = Id$. Since $j^*(d\omega) \in \Omega^n(\partial B^n)$, $j^*(d\omega) = 0$. Therefore, by Stokes' theorem,

$$\int_{\partial B^n} j^*(\omega) = \int_{B^n} d\big(j^*(\omega)\big) = \int_{B^n} j^*(d\omega) = 0, \tag{5.32}$$

which gives a contradiction. □

Next we show that we can find such a $\omega \in \Omega^n(B)$ with $\int_{\partial B^n} \omega > 0$.

Definition 5.1. $\omega \in \Omega^n(M^n)$ is called a **volume form** if $\omega_p(v_1, \cdots, v_n) > 0$ for every $p \in M^n$, where $v_1, \cdots, v_n$ is a positively oriented basis of T_pM^n.

Let M^n be compact, we claim that for any volume form ω, $\int_{M^n} \omega > 0$. By definition,

$$\int_{M^n} \omega = \int_{U_\alpha} \sum_\alpha (\phi_\alpha \omega), \tag{5.33}$$

where $\{\phi_\alpha\}$ is a partition of unity subordinate to charts $\{U_\alpha\}$ of M^n. Since $\phi_\alpha \geq 0$, $\phi_\alpha \omega_p(v_1, \cdots, v_n) \geq 0$ for every p. $\sum \phi_\alpha = 1$ implies that for any $p \in M^n$, there exists an α such that $\phi_\alpha(p) > 0$. So

$$\int_{U_\alpha} \phi_\alpha \omega > 0. \tag{5.34}$$

Note that $(x_\alpha^{-1})^*(\phi_\alpha \omega) = f dx_1 \wedge \cdots \wedge dx_n$, and we have $f(p) > 0$ and $f \geq 0$. Now, since $\sum_\beta \phi_\beta \omega \geq 0$ and $\int \phi_\alpha \omega > 0$, $\sum_\beta (\int_{U_\beta} \phi_\beta \omega) > 0$.

Proposition 5.5. *There exists a volume form $\omega \in \Omega^n(M^n)$ if and only if M^n is orientable.*

Proof. ($\Leftarrow$) Clear.
($\Rightarrow$) Given $\omega \in \Omega^n(M)$, define $v_1, \cdots, v_n$ to be oriented if $\omega(v_1, \cdots, v_n) > 0$. If $w_1, \cdots, w_n$ is oriented, and $v_i = Lw_i$, then $\det L > 0$. Since $L^*(\omega) = (\det L)\omega$,

$$L^*(\omega)(v_1, \cdots, v_n) = \omega(Lw_1, \cdots, Lw_n) = (\det L)\omega(w_1, \cdots, w_n). \tag{5.35}$$

Hence $\omega(w_1, \cdots, w_n) > 0$. □

Theorem 5.6. *Let M^n be a compact manifold with $\partial M \neq 0$, then there exists no map $j : M \to \partial M$ such that $j\big|_{\partial M} = Id$. (j is a retraction.)*

Proof of the Stokes' theorem

Proof. (a) Locally, $\omega \in \Omega^{n-1}(\mathbb{R}^n)$. Let $supp(\omega) \subset U$, where $U \in \mathbb{H}^n$ is an open set such that $\partial \bar{U}$ is compact. We can write $\omega = \sum_i f_i dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_n$, so

$$d\omega = \sum_i df_i \wedge dx_i \wedge dx_1 \wedge \cdots \wedge dx_n \tag{5.36}$$

$$= \sum_{i,j} \frac{\partial f_i}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge dx_n \tag{5.37}$$

$$= \sum_{i,j} \frac{\partial f_i}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge dx_n \tag{5.38}$$

(5.39)

□