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Math 600

Distribution theory

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Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

1 Test functions and distributions

2 Distributions of compact support

3 Tempered distributions and Fourier transforms

3.1 Functions with rapid decay

We need a new space of functions.

Definition 3.1 (Schwartz space)**.** The space $\mathcal{S}(\mathbb{R}^n)$ consists of all smooth functions $\phi : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|\phi\|_{\alpha,\beta} := \sup_x |x^\beta D^\alpha \phi(x)| < \infty \tag{3.1}$$

for multi-indices α, β. Note that $\|\cdot\|_{\alpha,\beta}$ are seminorms. We say a sequence $\{\phi_m\}_{m\geq 1}$ tends to 0 in $\mathcal{S}(\mathbb{R}^d)$ if $\|\phi_m\|_{\alpha,\beta} \to 0$ for each α, β.

A Typical Schwartz function is the Gaussian $\phi(x) = e^{-|x|^2}$. We know that ϕ and its derivatives tend to zero faster than any $(1 + |x|^2)^{-N}$ for any $N \geq 0$.

Definition 3.2. A linear map $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ belongs to $\mathcal{S}'(\mathbb{R}^n)$, the space of **tempered distributions**, if there exists constants C, N such that

$$| < u, \phi > | \leq_{|\alpha|,|\beta|\leq N} C\|\phi\|_{\alpha,\beta} \tag{3.2}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Exercise 3.1. Provide an equivalent definition using the sequential continuity.

3.2 Fourier transform

We have the following inclusion of spaces: $D(\mathbb{R}_n) \hookrightarrow \mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{E}(\mathbb{R}^n)$, and conversely $\mathcal{E}'(\mathbb{R}_n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \hookrightarrow D'(\mathbb{R}^n)$.
Recall the **Fourier transform** of an integrable function $\phi \in L^1(\mathbb{R}^d)$ is defined by

$$\hat{\phi}(\lambda) = \int e^{-i\lambda\cdot x}\phi(x)dx. \tag{3.3}$$

Lemma 3.1. *If $\phi \in L^1(\mathbb{R}^n)$, then $\phi \in C(\mathbb{R}^n)$.*

Proof. Suppose $\{\lambda_m\}_{m\geq 1}$ is a sequence such that $\lambda_m \to \lambda$ in $\mathbb{R}^n$, then

$$\lim_{m\to\infty} \hat{\phi}(\lambda_m) = \lim_{m\to\infty} \int e^{-i\lambda_m\cdot x}\phi(x)dx \tag{3.4}$$

$$= \int \lim_{m\to\infty} e^{-i\lambda_m\cdot x}\phi(x)dx \tag{3.5}$$

$$= \int \lim_{m\to\infty} e^{-i\lambda\cdot x}\phi(x)dx = \hat{\phi}(\lambda) \tag{3.6}$$

where we have used the dominate convergent theorem in the second equality since $|e^{-i\lambda_m\cdot x}\phi(x)| \leq |\phi(x)| \in L^1(\mathbb{R}^n)$. Hence $\hat{\phi}$ is continuous. □

Remark 3.1. $\hat{\phi}$ is bounded since $|\hat{\phi}| \leq \int |\phi(x)|dx$.

Definition 3.3. For $\phi \in L^1(\mathbb{R}^n)$, define the Fourier transform of ϕ by setting $(\hat{\phi})(\lambda) = \int e^{-i\lambda\cdot x}\phi(x)dx$.

Note that if $\phi \in \mathcal{S}(\mathbb{R}^n)$, then $\phi \in L^1(\mathbb{R}^n)$. Indeed,

$$\int |\phi(x)|dx = \int (1+|x|)^{-N}(1+|x|)^N|\phi(x)|dx \tag{3.7}$$

$$\leq C \sum_{|\alpha|\leq N} \|\phi\|_{\alpha,0} \int (1+|x|)^{-N}dx < \infty \tag{3.8}$$

for N sufficiently large.
So if $\phi \in \mathcal{S}(\mathbb{R}^n)$, from lemma, we know that ϕ is continuous. In fact, we can say more.

Lemma 3.2. *For $\phi \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\widehat{(D^\alpha\phi)} = \lambda^\alpha\hat{\phi}(\lambda) \tag{3.9}$$

$$\left(\widehat{x^\beta\phi(x)}\right) = (-D_\lambda)^\beta\hat{\phi}(\lambda) \tag{3.10}$$

for each multi-indices α, β.

Note that $D = i\partial$.

Proof. Since $|x^\alpha D^\beta\phi| \to 0$ rapidly as $|x| \to \infty$ for all α, β. Using integration by parts,

$$\widehat{(D^\alpha\phi)}(\lambda) = \int e^{-i\lambda\cdot x}D_x^\alpha\phi(x) \tag{3.11}$$

□