CHAPTER 1

Introduction

1. What is a PDE?

1.1. In words... A partial differential equation is a *functional equation*, where the unknown is a function, and the rigorous setting is provided by functional analysis. It involves differential (and integral) operators, which can be seen as infinite-dimensional counterparts of matrices, and therefore requires knowledge from *differential calculus* and *spectral theory*. PDEs are ubiquitous in almost all areas of modern science (in various areas in pure and applied mathematics of course, but also in physics, engineering, biology, economy...)

1.2. In equations... Let us consider a function $\mathcal{F} = \mathcal{F}(x, y_1, \dots, y_N)$ depending on $n+1$ real variables, one can search for a differentiable function $u = u(x)$ such that its successive derivatives $u, u', \dots, u^{(n)}$ satisfy the implicit equation

$$\forall\, x \in \Omega, \quad \mathcal{F}\left(x, u(x), u'(x), \dots, u^{(n)}(x)\right) = 0$$

where $\Omega \subset \mathbb{R}$ is the domain (i.e. open connected regular set) to be made precise. The study of this general question is the object of the theory of **ordinary differential equations** (ODEs).

The fundamental problem in the theory of **partial differential equations** (PDEs) differs by the fact that one considers unknown functions $u = u(x_0, x_1, \dots, x_\ell)$, $\ell \geq 1$, which *depend on several real variables*. Then the differential relation involves the different *partial derivatives* of u:

$$\frac{\partial u}{\partial x_i}, \ \frac{\partial^2 u}{\partial x_i \partial x_j}, \ \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_l}, \dots$$

The corresponding general question of this theory now becomes (in a fuzzy – because general – formulation): let us consider a function

$$\mathcal{F} = \mathcal{F}(x_0, x_1, \dots, x_\ell, y_0, \dots, y_d, y_{00}, \dots, y_{\ell\ell}, \dots)$$

with a certain finite number of real variables, then one searches for $u = u(x_0, x_1, \dots, x_\ell) : \mathbb{R}^{\ell+1} \to \mathbb{R}$ that satisfies

$$\text{(1.1)} \quad \forall\, x_0, x_1, \dots, x_\ell \in \Omega,$$
$$\mathcal{F}\left(x_0, x_1, \dots, x_d, \frac{\partial u}{\partial x_0}, \dots, \frac{\partial^2 u}{\partial x_i \partial x_j}, \dots, \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_l}, \dots\right) = 0$$

where $\Omega \subset \mathbb{R}^{1+\ell}$ is a domain to be made precise. Such a function u is called **a solution to the PDE** (1.1). When possible (for instance, not for elliptic PDEs!), it will pay in general to identify the $x_0 = t$ variable with time and the other variables with space coordinates or phase space coordinates (position and velocity) in physical problems. This can be in itself a source of difficulty as in general relativity.

Observe that in general the solution depends again on "parameters" prescribed by limit conditions. The situation is however more complicated than before. Assuming that the highest derivatives on the $x_0 = t$ variable is of order one, with a constant coefficient, then the limit condition now corresponds to prescribing the values of u on the hypersurface $\mathbb{R}^\ell \sim (0, x_1, \dots, x_\ell) \subset \mathbb{R}^{1+\ell}$. More generally for first-order differential operators, the hypersurface should satisfy some geometric *non-degeneracy conditions* (see later in Chapter 2).

1.3. PDEs as abstract generalisation of ODEs? If we try to look at PDEs with "ODE eyes" and focus either on the dimension of the variable space, or on the dimension of the space where the unknown is valued we obtain the two following abstract viewpoints:

(1) either generalization of ODEs for more than one variable, i.e. more than one dimension for the space of parameters (which leads to partial derivatives).
(2) or, when one of the variables can be identified as "time", as ODEs in infinite dimension.

Let us expand a bit more on the viewpoint (2) in the case when a time variable can be identified. The is a conceptual jump from a scalar valued trajectory to the "trajectory" of a function of the other "space" variables. That is, denoting again $x_0 = t$ and assuming that we can "resolve" in the first time derivative, we can write the PDE problem on $u(t, x_1, \dots, x_\ell)$ as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathfrak{G}(u) := \mathcal{G}\left(u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, \dots\right)$$

where $\mathfrak{G}$ is an abstract nonlinear operator on a functional space $u \in \mathcal{E}$, and $\mathcal{G}$ is a scalar function depending on a finite number of real variables. From this viewpoint it appears clearly that *PDEs are nothing but infinite-dimensional ODEs*, with trajectories in infinite-dimensional functional spaces rather than in $\mathbb{R}$ or $\mathbb{R}^m$. Hence at a very abstract level the theory of PDEs could be considered as a sub-branch of *infinite-dimensional dynamical systems.*

Each of these two abstract viewpoints enlighten a fundamental novel difficulty of PDEs as compared to ODEs:

(1) As one abandons the total order of the real line for the space of parameters, new deep geometric phenomena appear (hyperbolicity-ellipticity-parabolicity…), which corresponds to fundamental questions of physical relevance: Can we parametrise a variable as time, and if yes, is the evolution reversible, or at least solvable in both directions of time, or not? Does it have finite or infinite speed of propagation? Related to this, the boundary conditions can be *characteristic* and show non-trivial geometric issues, as we shall discuss in Chapter 2. [1]

[1] Note that somehow another similar conceptual gap exists between *scalar PDEs* (for which the total order of the real line can be used on the values taken by the unknown) and systems of PDEs,

(2) Since norms are no more equivalent and are infinitely many in infinite dimension, the controls of (even linear!) operators is not for granted and the choice-construction of suitable norms for estimating solutions becomes a core conceptual difficulty. The question of "which regularity and which decay" to choose is key in the study of PDEs; the functional space chosen can be thought of as the "microscope" of the PDE analyst.

However these viewpoints do not bring any new information per se, as the theorems of ODEs do not apply, but they are inspirative for the intuition (and in the linear case the viewpoint (2) motivated a specific field of mathematics, the infinite-dimensional generalization of the theory of matrices, the *semigroup theories*).

Last but not least, the nonlinearity becomes *much* harder to understand in a PDE context. This cannot be reduced to the scalar comparison with the linear case (superlinear, sublinear) together with accounting for the sign, as we shall discuss later in the course.

1.4. PDE as a unified field? Combined together these two facets mean that a *general systematic theory*, such as we have for ODEs, is **not** possible for PDEs. The particular structure of each equation at hand has to be understood and used in the analysis. In this sense the rigidity of the problem at hand has to respected in the analysis of PDEs, and it is of crucial importance to focus on the **fundamental equations**[2], or, when the problems look too hard, to devise simplified models which carefully capture some important structural aspects. One should not "cook" models at random – this would result in intractable or trivial-useless equations with probability 1! – or cook models in order to adapt existing tools. Nevertheless there are deep unifying concepts, ideas and goals that we will try to explain, allowing still to speak of PDEs as a unified field. This field enjoys a crossroad position between analysis, geometry, probability and numerical analysis.

1.5. Some bibliographical references. The useful prerequisites are: basics in ODE theory, measure theory, Fourier analysis, matrix theory and linear algebra, linear analysis (Hilbert spaces, Banach spaces). Apart from these basic knowledges the course will be self-contained and we will try to briefly introduce all the objects needed, including some recalls on these basic notions.

Here are some references for the analysis tools used in this course:

- Lieb, E. and Loss, M. *Analysis* [**10**].
- Brézis, H. *Functional Analysis* [**1**].
- Rudin, W. *Functional Analysis* [**12**].
- The core Part III courses in Analysis and some in Applied Analysis.

Here are some more specialized references in PDEs:

- Evans, C. *Partial differential equations* [**6**].
- John, F. *Partial differential equations* [**8**].

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and it is related to deep difficulties and open questions in systems of conservation laws, Einstein's equations, 3D Navier-Stokes equations...

[2]In the sense of being derived from first principles in physics, or at least enjoying a consensus as broad as possible among physicists.

- Strauss, W. *Introduction to partial differential equations* [**14**].
- Taylor, M. *Partial differential equations, Vol. I. Basic theory* [**15**].
- Rauch, J. *Partial differential equations*, [**11**].
- Courant, R. and Hilbert, D. *Methods of Mathematical Physics* (2 volumes) [**3, 4**]
- The specialised Part III courses in Analysis or Applied Analysis…

1.6. Structure of the course. The "lecture" part of the course will be self-contained and contain all the required knowledge for part III students. In these lectures we will first introduce the core concepts by studying carefully simple instances of each of the main linear equations, emphasizing each time the methods. Finally we will conclude with a discussion of a few paradigmatic problems in nonlinear PDEs analysis and try to give a flavor of the geography of this immense field of research. The style of the lectures part of the course will be more brief than a textbook treatment of some of the material (see the bibliography) and in particular will not typically aim at the sharpest results. On the other hand, some material will be emphasized that is not often concisely presented in textbooks. These lectures will be complemented by four example classes and a revision class.

The other "presentation" part of the course will be compulsory only for CCA students. It will consist first in a mid-term assignement of studying four important results in linear analysis that are important in PDEs and present in some afternoon sessions to the class. The part III students are of course most welcome to these sessions, but the material covered there will **not** be examinable. Second CCA students will prepare a final assignement in four groups again, each one focusing on the study of the mathematical theory behind an important nonlinear PDE of mathematical physics or geometry. The group work will in general be the reading of one or two difficult research paper, with the mentoring help of a faculty member of the CCA. This will result in afternoon presentations to the CCA cohort in early january. Again interested Part III students are most welcome but this is not examinable.

The starting point for any analysis of nonlinear PDE is of course linear PDE, and the pedestrian explanation for this is simply that one understands a lot by what is essentially linearisation. A more subtle reason is that almost invariably hard problems in nonlinear PDEs require solving specific problems in linear PDEs (through iteration and so on…) A nonlinearity conceptually corresponds to a *causality loop*, and our only way so far to approach mathematically such loops is to first break it (through one way or another of linearisation and iterative or bootstrap scheme) and then reconstruct it by asymptotic convergence. On the other hand, one can certainly know too much "linear PDEs" for ones own good.

The larger part of the course will thus be devoted to the understanding of basic paradigmatic linear PDEs. However we will introduce PDEs in this chapter by building on the intuition students have learned in ODEs and will try to make a smooth transition between the two concepts. We will accordingly begin our journey into PDEs with the only systematic theorem that reminds the Cauchy-Lipschitz (Picard-Lindelöf) theorem in ODEs, the Cauchy-Kovalevskaya theorem. This will also allow us to introduce some core geometric concepts explaining the classification of linear PDEs used in the next chapters.

- Chapter 1: Introduction (From ODEs to PDEs)
- Chapter 2: The Cauchy-Kovalevskaya theorem
- Chapter 3: Ellipticity (Laplace equation, Poisson equation, heat equation)
- Chapter 4: Hyperbolicity (transport equation, wave equation)
- Conclusion: Nonlinearity, open problems

2. The Cauchy problem for ODEs

2.1. "Solving the equations". School education teaches us that equations are something which you "solve". The quicker one unlearns this idea, the better. In the antideluvian world, indeed, to do PDEs meant to explicitly solve the equations. The revolution in point of view was that one could understand solutions without being able to write them down explicitly in closed form. This is of course nothing other than the final round of the two-thousand year old revolution which defines what we call today Analysis, see more in the historical notes.

2.2. The theory of integration. One of the most important question in the birth of modern mathematical analysis is the "inversion" of the process of the differentiation. This was a key motivation to Leibniz and Newton's theories of differential calculus.[3]

The basic example is the following: let a function $F : \mathbb{R} \to \mathbb{R}$ with some regularity (say continuous) and one searches for the functions $u : \mathbb{R} \to \mathbb{R}$ differentiable on $\mathbb{R}$ and such that

$$u'(t) = F(t), \quad t \in I \tag{2.1}$$

where I is an open interval of $\mathbb{R}$. In this very simple case, the answer to this question is provided by *the fundamental theorem of integration* which writes

$$u(t) = \int_{t_0}^{t} F(y)\,\mathrm{d}y + u(t_0), \quad t_0 \in \mathbb{R}.$$

This is a one-real-parameter family, and this parameter is determined by the *limit condition* $u(t_0)$ at point t_0.

Exercise 1. *What is the minimal regularity of F for which you know how to give meaning to the previous formula (and therefore find the solutions to* (2.1)*)? Show for instance that it is enough that f admits everywhere right and left limits (Riemann integral theory), or even more generally that f is Lebesgue-integrable (Lebesgue-Borel integral theory). In these cases, is the function u differentiable everywhere? (build a counter-example).*

This allows hence to find a (and in fact *the*) solution that takes a given value u_0 at t_0. We have therefore solved the following three mathematical questions:

- Find a *particular* solution.
- Find *all possible* solutions.
- Find *the* solution that satisfies certain limit conditions.

[3]In the words of this time, "the problem of finding systematically the tangents and the surface under a general curve" – and not only for polynomials for which empirical formula had been derived.

Remark 2.1. *Remember that in both theories of integration (Riemann and Lebesgues-Borel) the integral is defined as a* limit*, based on monotonous convergence and Cauchy criterion. Hence the completeness of* $\mathbb{R}$ *is crucially used here, and also some numerical algorithm can be deduced from the proof in order to calculate the integral, it is already* ***not*** *accessible in general through an explicit formula.*

2.3. The general theorems. The starting point of the theory of ODEs is when the RHS in (2.1) depends on the solution u in a *local manner*, i.e. $F = F(t, u(t))$ with F defined on $I \times \mathbb{R}$, which leads to

$$u'(t) = F(t, u(t)), \quad t \in I \subset \mathbb{R}. \tag{2.2}$$

We consider a *system* of m coupled equations which write just like (2.2), but with a vectorial $\mathbf{u} = (u_1, \ldots, u_m)$ unknown function:

$$\mathbf{u}'(t) = \mathbf{F}(t, \mathbf{u}(t)), \quad t \in I \subset \mathbb{R} \tag{2.3}$$

where now $\mathbf{F} = (F_1, \ldots, F_m)$ is a vectorial function, defined on $I \times \mathbb{R}^m$. This means in a more explicit form

$$\begin{cases} u_1' = F_1(t, u_1, \ldots, u_m) \\ u_2' = F_2(t, u_1, \ldots, u_m) \\ \ldots \\ u_m' = F_m(t, u_1, \ldots, u_m) \end{cases} \tag{2.4}$$

and $\mathbf{F}$ is called the *vector field* of the ODE. A solution to this system of ODEs is often called a *flow*. The pair $(t, \mathbf{u})$ belongs to $I \times \mathbb{R}^m$, on which $\mathbf{F}$ is defined.

Remark 2.2. *Recall how a m-th order scalar differential equation in the form* $u^{(m)}(t) = F(t, u) \in \mathbb{R}$ *can be reduced to* (2.4)*.*

There are then essentially[4] three key results, ranked in a decreasing way according to the level of regularity assumed on the vector field. The first result has a huge theoretical importance, but is not very useful in practice. It however corresponds to the first historical attempt of "solving explicitely" nonlinear ODEs, and the methodology has transformed and survived in many areas of PDE analysis. We do not give full technical details in the statement, but we will do so for the next one that has been studied in undergrad.

Theorem 2.3 (Cauchy-Kovalevskaya theorem for ODEs[5]). *In the (open) region* $\mathcal{A} \subset I \times \mathbb{R}^m$ *where the vector field* $\mathbf{F}$ *is (real)-analytic*[6] *according to both its arguments, there exists a unique local analytic solution: for any* $(t_0, \mathbf{u_0}) \in \mathcal{A}$*, there is a neighborhood* $\mathcal{V}(t_0) \times \mathcal{V}(\mathbf{u}_0) \subset \mathcal{L}$ *so that the ODE has a unique local real analytic solution in this neighborhood that satisfies* $\mathbf{u}(t_0) = x_0$*.*

[4]There is also an important more recent result, the DiPerna-Lions theorem [**5**], showing some existence and uniqueness to ODE in an *almost sure sense*, under for instance the assumption $\mathbf{F} = \mathbf{F}(\mathbf{u}) \in W^{1,1}$, $\nabla \cdot \mathbf{F} = 0$.

[5]First proved by A. Cauchy for first-order quasilinear PDEs in 1842 and then improved into its modern form by S. Kovalevskaya in 1875.

[6]Let us recall that a real function is said to be *analytic* at a given point if it possesses derivatives of all orders and agrees with its Taylor series in a neighbourhood of the point.

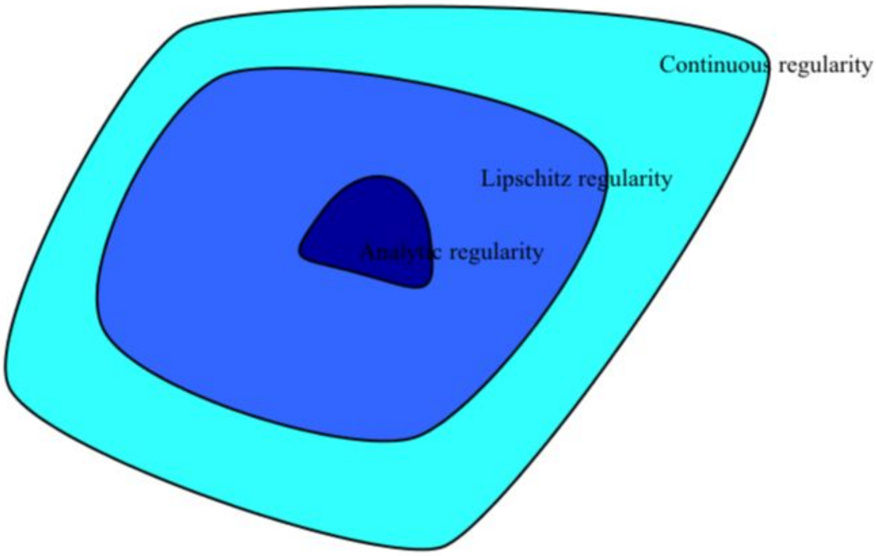

FIGURE 2.1. Regularity of the force field $\mathbf{F}$ in the different regions $\mathcal{A} \subset \mathcal{L} \subset \mathcal{C} \subset I \times \mathbb{R}$ (with $m = 1$).

The second result is by far the most used practically, because it still provides local existence and uniqueness under a much lower regularity assumption on the vector field $\mathbf{F}$.

THEOREM 2.4 (Cauchy-Lipschitz / Picard-Lindelöf theorem[7]). *In the (open) region $\mathcal{L} \subset I \times \mathbb{R}^m$ where $F = F(t, \mathbf{u})$ is continuous in both variables and locally Lipschitz according to the second argument, there is local existence and uniqueness of solutions, i.e. for any $(t_0, \mathbf{u_0}) \in \mathcal{L}$, there is a neighborhood $\mathcal{V}(t_0) \times \mathcal{V}(\mathbf{u_0}) \subset \mathcal{L}$ so that the ODE has a unique local solution in this neighborhood that satisfies $\mathbf{u}(t_0) = x_0$. These solutions are C^1 and form (locally) a m-parameters family which depends continuously on $\mathbf{u_0} = (u_1(t_0), \ldots, u_m(t_0))$.*

REMARK 2.5. *The definitionof F being locally Lipschitz according to the second argument is: for any $(t_0, u_0) \in \mathcal{L}$, there is a neighborhood $\mathcal{V}(t_0) \times \mathcal{V}(\mathbf{u}_0) \subset \mathcal{L}$ and a constant $C > 0$ (depending on the neighborhood) such that*

$$\forall\, t \in \mathcal{V}(t_0),\ \forall\, \mathbf{u}, \mathbf{v} \in \mathcal{V}(\mathbf{u}_0), \quad |F(t, \mathbf{u}) - F(t, \mathbf{v})| \leq C|\mathbf{u} - \mathbf{v}|.$$

REMARK 2.6. *The modern version of the proof of this theorem is based on Picard's fixed point theorem: the methodology survives in many proofs of construction of local-in-time solutions in PDEs.*

Finally the third and last theorem deals with a weaker regularity, which is natural for the existence of solutions, the mere continuity of the vector field $\mathbf{F}$. The flow is then constructed by an approximation scheme via the tangents to the curve, but there is no

[7]Appearing first in the course of Cauchy at École Polytechnique in the 1830's, then improved in its modern form by Lipschitz, Picard and Lindelöf.

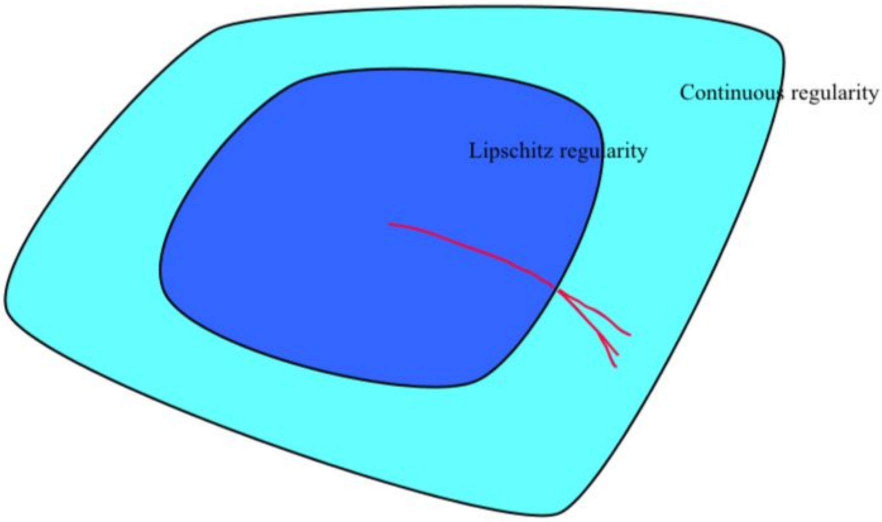

FIGURE 2.2. Non-uniqueness when the solution leaves $\mathcal{L}$.

uniqueness property in general. An interesting point is that this proof is a prototype of many proofs of construction of (weak) solutions by approximation and compactness in PDEs.

THEOREM 2.7 (Cauchy-Peano's theorem[8]). *In the region $\mathcal{C} \subset I \times \mathbb{R}^m$ where the vector field $\mathbf{F}$ is continuous in both its arguments, there is local existence of C^1 solutions (i.e. for any $(t_0, \mathbf{u_0}) \in \mathcal{C}$ in the phase space, there is a local solution in the neighbourhood of this point).*

EXERCISE 2. *See the example sheet for a proof of this theorem, based on an approximation scheme and the Arzelà-Ascoli compactness theorem.*

Let us give a visual interpretation of these results. In the regions $\mathcal{A}$ and $\mathcal{L}$ the solutions are unique and locally partition the regions (as implied by the uniqueness property). The analytic regularity breaks when they leave $\mathcal{A}$. When they reach the frontier of $\mathcal{L}$ (still within $\mathcal{C}$), the flow may separate into several solutions (out of the same tangent). Observe that the "frontier" of $\mathcal{L}$ can be at infinity.

REMARKS 2.8.
- *For the solution to be able to "leave" $\mathcal{L}$, one needs that the vector field $\mathbf{F}(t, \mathbf{u})$ is nonlinear in the second variable. Indeed observe that if it is linear in $\mathbf{u}$ then $\mathcal{L} = I \times \mathbb{R}^m$.*
- *This gives a first illustration of the importance of nonlinearity in the question of uniqueness. A typical example is*

$$\begin{cases} u'(t) = \sqrt{u(t)}, \quad t \in (-a, a), \; a > 0 \\ u(0) = 0. \end{cases} \tag{2.5}$$

[8]Published in 1890 by G. Peano as an extension of A. Cauchy's theorem

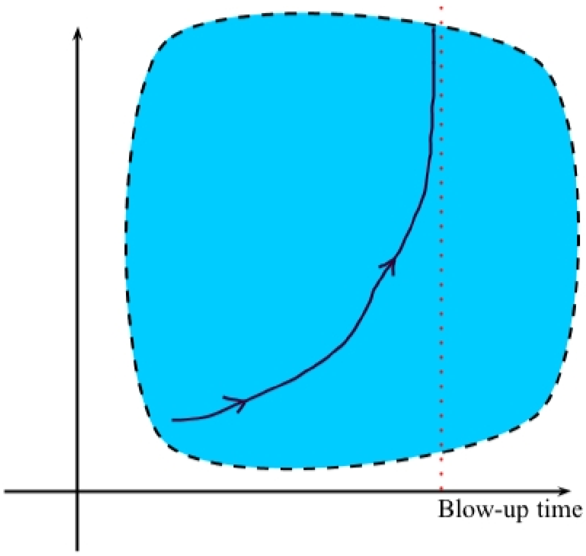

FIGURE 2.3. No global continuation as the solution goes to infinity in finite time.

- *Here are the two examples given by G. Peano in his 1890 paper:*

$$\begin{cases} u'(t) = 3t^{2/3}, \quad t \in (-a, a), \ a > 0 \\ u(0) = 0. \end{cases} \tag{2.6}$$

and

$$\begin{cases} u'(t) = \frac{4u(t)t}{u(t)^2+t^2}, \quad t \in (-a, a), \ a > 0 \\ u(0) = 0. \end{cases} \tag{2.7}$$

EXERCISE 3. *Show that solutions to linear ODEs are global in time. Show that the previous three nonlinear equations admit infinitely many solutions. More precisely in the first two cases show that there are infinitely many solutions splitting into two different types, and in the third case show that they split into five* different types *of solutions.*

2.4. Local *vs.* global solutions. So far, we have discussed only *local* phenomena (existence, regularity and uniqueness of the flow close to a given point in the phase space), according to the regularity of the vector field $\mathbf{F}$. Let us now discuss another question, that of how far the solutions constructed so far locally can be continued w.r.t. the variable t. In case they are continued on I, we speak of *global solutions*.

As clear from the pictures above, even when the frontier of $\mathcal{L}$ and $\mathcal{C}$ is at infinite, one should pay attention to *how fast* the solution reaches this frontier in terms of the variable x, and it can happen *in finite time* leading to a non-global solution. We hence see a fundamental phenomenon, which also will appear and play a key role in PDEs:

differential evolution system for which the solution cannot be continued after a certain time as it becomes infinite.[9]

A paradigmatic example is

$$\begin{cases} u'(t) = u(t)^2, & t \in \mathbb{R}, \\ u(0) = u_0 > 0. \end{cases} \tag{2.8}$$

EXERCISE 4. *Show this phenomenon on the previous example.*

However the following example, also with the same order of nonlinearity, exhibits a radically different behavior:

$$\begin{cases} u'(t) = -u(t)^2, & t \in \mathbb{R}, \\ u(0) = u_0 > 0. \end{cases} \tag{2.9}$$

EXERCISE 5 (See the first example sheet). *Show that there are global solutions to the previous equation. Show also that the value $u(1)$ solution at $t = 1$ is bounded by a bound* independent *of the value of $u_0 \geq 1$ (study the explicit solution). What about the case $u_0 = +\infty$? This phenomenon of* appearance of new estimates independently of the initial data *will be encountered again in a new form for parabolic PDEs.*

The two last examples illustrate that fact that for nonlinear ODEs, the *sign* of the nonlinearity matters even for simply continuing the solution for large times. The importance of the sign will still play a dramatic role for PDEs, however nonlinearity will be much more complicated to understand.

The simplest criterion which is taught in ODE courses in order to avoid this "blow-up" is the uniform Lipschitz bound. However we will see that such a criterion is most of the time unexportable to PDEs (even for linear PDEs), due to the presence of unbounded operators. However another criterion exists, based on estimating bounds along time on the flow. It is interesting to discuss it in detail as it corresponds to the intuition of the "a priori bounds" in PDEs.

Consider $I = \mathbb{R}$ and $\mathbf{F}$ a C^1 vector field on $I \times \mathbb{R}^m$ such that

$$\forall t \in \mathbb{R},\ \mathbf{u} \in \mathbb{R}^m, \quad |\mathbf{F}(t, \mathbf{u})| \leq C(1 + |\mathbf{u}|)$$

for some constant $C > 0$, and the following ODE

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{F}(t, \mathbf{u}(t)), \quad t \in \mathbb{R}.$$

EXERCISE 6. *(a) Show that the solutions are global. (Hint: Use the theorem of "leaving any compact set".)*

(b) Use this result to show that the solutions to the pendulum equation $u'' + \sin u = 0$, $u(0) = u_0 \in \mathbb{R}$, are defined globally.

(c) Use this result to construct global solutions to the following equation: $u'' + \sin u^2 = 0$, $u(0) = u_0 \in \mathbb{R}$. Check that in this case the uniform Lischitz bound fails.

[9]It is customary to use the vocabulary of a "time" variable for the variable of the ODE.

2.5. A gallery of important examples.

(1) The *linear* system of differential equations

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = A\,\mathbf{u}(t), \quad t \in \mathbb{R},\ \mathbf{u}(t) \in \mathbb{R}^m,\ A \in \mathcal{M}_{m,m}(\mathbb{R}) \tag{2.10}$$

whose (in)-stability properties are encoded in the spectral properties of the matrix A. It is all the more important to perfectly understand the properties of this linear differential systems as it naturally appears in *linearization* study of nonlinear differential system. An over-simplified example is the linear pendulum $u''(t) = -u(t)$ obtained by linearizing the correct equation from physics $u''(t) = -\sin u(t)$.

Exercise 7. *Some recalls on ODEs...*

- *Assume that A can be diagonalised. What is the behavior of $\mathbf{u}(t)$ as $t \to \pm\infty$ depending on the spectrum of A?*
- *What about the general case when A cannot be diagonalised (use Jordan's decomposition)?*
- *Search (on the internet or in the library) for the Hartman–Grobman theorem and the Poincaré–Bendixon theorem.*

(2) For nonlinear ODEs, the two paradigmatic examples (already encountered) to always keep in mind are

$$u'(t) = \sqrt{u(t)} \quad \text{and} \quad u'(t) = u(t)^2.$$

This highlights the important guiding principle that sublinearities typically can create non-uniqueness and superlinearities typically can create blow-up in finite time, that will still be useful for PDEs. However proof of uniqueness in PDEs will be in general much harder again due to the presence of unbounded operators and the many possible topologies and norms.

(3) The *logistic equation*[10]

$$u'(t) = ru(t)\left(1 - \frac{u(t)}{k}\right), \quad r, k \in \mathbb{R}, \tag{2.11}$$

is one of the first historical example of nonlinear ODE, whose mathematical structure is extremely rich (logistic application and sequence, onset of chaos and bifurcation in dynamical systems, and Charkovski's theorem of 1964 "3-cycles imply chaos"...).

3. The Cauchy problems for PDEs

3.1. The notion of Cauchy problem. Let us first fix some notation. Assume that we are in a situation where we can particularize one of the variables in the PDE as time t (it is not possible for elliptic PDEs for instance...), and consider it again as an evolution problem. Assume that we are in a situation where the case with m-th order

[10] Sometimes also called the *Verhulst model*, in the name of the Belgian mathematican Pierre Verhulst (1804–1849). It was derived by Verhulst in population dynamics in opposition to Malthus' model of indefinite geometric growth $u'(t) = ru(t)$. It was rediscovered by Pearl in 1920 and Volterra in 1925.

derivatives in time can be reduced to a system of m PDEs with first order derivatives in time, with the simple form

$$\text{(3.1)} \qquad \begin{cases} \partial_t \mathbf{u} = \mathbf{F}\left(t, x_1, \ldots, x_\ell, \partial_1 \mathbf{u}, \ldots, \partial_\ell \mathbf{u}, \partial_{ij}^2 \mathbf{u}, \ldots\right) =: \mathfrak{G}(t, \mathbf{u}(t)), \\ \mathbf{u} = \mathbf{u}(t, x_1, \ldots, x_\ell) \in \mathbb{R}^m. \end{cases}$$

The question of building solution to this equation is the first one may ask mathematically on a PDE. The index ℓ is the dimension of the problem (number of parameters of the physical system), and m is the number of coupled equations (number of physical quantities whose evolution is modeled in the equation).

If the vector field $\mathbf{F}$ does not depend on time t, the PDE is said to be *autonomous* as for ODEs. Similarly a solution $\mathbf{u}$ which does not depend on time is said to be *stationary*. The variables $(t, x_1, \ldots, x_\ell) \in \Omega$ belong to a space-time domain Ω. *Space-time boundary conditions* are prescribed on the boundary of the domain $\partial\Omega$, in the form of some given function $\mathbf{u}_b$ on this submanifold. In the important particular case where $\Omega = I \times \Omega_{\mathbf{x}}$ where the domain in space does not depend on time, we distinguish *initial conditions* on $\{0\} \times \Omega_{\mathbf{x}}$ and *boundary conditions* on $I \times \partial\Omega_{\mathbf{x}}$. A problem can be *under-determined* or *over-determined* according to these space-time boundary conditions.

The given of a PDE (3.1) together space-time boundary conditions is called a **Cauchy problem**. As for ODEs in Picard-Lindelöf's theorem we might ask about *existence*, *uniqueness*, and *continuous dependence according to the prescribed parameters of the problem.* This results into

3.2. The notion of well-posedness.

Definition 3.1 (Well-posedness). *A Cauchy problem is* well-posed (in the sense of Hadamard) *if*

(1) there exists a solution,
(2) this solution is unique,
(3) the solution depends continuously on the boundary conditions (in a reasonable topology).

This definition was formalized by J. Hadamard in the paper "*Sur les problèmes aux dérivées partielles et leur signification physique*" in 1902, as an attempt to clarify the link between PDE analysis and physics. Indeed, as soon as one abandons the "old world" of explicit solutions, adopting the viewpoint of Cauchy of constructing solutions through approximation, iterations, fixed-points, etc. using the completeness of the real line and modern analysis, then ensuring that solutions are associated in a proper sense to a PDE (viewed as a relation between partial derivatives between some observables quantities) is crucial for checking the minimal consistency of the mathematical model with the real phenomena it is meant to describe.

The *existence* is an obvious requirement and relates to the non overdeterminacy of the model, the *uniqueness* relates to non underdeterminacy of the model. Together they relate to the *causality principle*: the future can be determined as caused by the present. The last (and the most subtle) third point relates to our experience that causality behaves in a continuous way (even when approaching threshold-bifurcation…) and that solutions need some stability according to the conditions giving rise to them in order to be "observable". Moreover in modelling of real world phenomena, the problem data always have some measurement or computational error in it, so without well-posedness, we cannot say that the solution corresponding to imprecise data is anywhere near the solution we are trying to capture. Thus, a necessary condition for a physics theory to have any predictive power is that it must produce well-posed problems. The concept of "well-posedness" has proved to be very useful in revealing the true nature of the equations, especially in identifying the "correct" initial and/or boundary conditions.

REMARK 3.2. *Sometimes the "correct" topology for the functional setting is suggested by the structure of the problem itself and physics (energy, entropy, etc.).*

3.3. What can we learn from the previous ODE results. Let us consider the previous results seen on ODEs and there possible extension to PDEs.

3.3.1. *The Cauchy-Kovalevskaya theorem.* There is a PDE version (in the more general form this is S. Kovalevskaya's contribution in the theorem). However (1) it requires analyticity of the function $\mathbf{F}$, (2) it is local in nature, (3) it has some limitations of its own for PDEs, namely $\mathbf{F}$ must involve only derivatives up to order 1. Nevertheless the methodology has survived in many subfields of PDEs.

EXERCISE 8. *Consider the following counter-example due to S. Kovalevskaya:*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u = u(t,x) \quad \textit{with} \quad t \in \mathbb{R},\ x \in \mathbb{R}$$

with the initial condition

$$u(0,x) = \frac{1}{1+x^2}$$

and show that the unique entire series solving this equation is divergent for any $t > 0$.

3.3.2. *The Picard-Lindelöf theorem.* The situation looks better. The proof relies on the fixed point theorem, which holds more generally in Banach spaces (complete normed vector spaces). However, to apply it, we need some boundedness of the operator $\mathfrak{G}$ on this space (when viewing the PDE as an infinite-dimensional ODE). There are particular cases where $\mathfrak{G}$ is a bounded multiplication or integral operator where it can be used, but typically if $\mathfrak{G}$ is some sort of differential operator, it is unbounded on most spaces one could think of. This hints that (keeping in mind the ODE examples of nonlinearities) it is compulsory in PDE to exploit the exact structure: which is what we generally call conservation laws (energy conservation, etc.), Liapunov (entropy) functionals, contraction properties, etc. Since one needs to understand this structure before actually having constructed the solutions, we call this stage "to devise *a priori estimates* on a PDE", the "a priori" means without still knowing beforehand whether we can construct solutions, but working at a purely formal level. This is a key aspect of PDE analysis.

REMARK 3.3. *Note that there is a kind of extension of Picard-Lindelöf theorem in PDE for unbounded* linear *operators which is called Hille-Yosida (which we will discuss in the mid-term CCA presentations). It relies on (1) an accretivity property (which is intuitively a "sign" of the operator), (2) the reduction to the case of bounded operator by an approximation (the Yosida regularisation), in order to use Picard-Lindelöf theorem.*

3.3.3. *The Cauchy-Peano theorem.* The proof relies on approximation proceedures and compactness arguments (Arzéla-Ascoli theorem). It in fact reminiscent of approximation-discretization proceedures in PDE for constructing solutions. This is usually much more involved for PDEs, but many important results of existence of "weak" (as opposed to regular) solutions roughly follow this strategy.

3.3.4. *Provisional conclusions.* Let us now draw an important lesson out of this discussion, which should be a guide for intuition in PDEs:

- The smaller (i.e., roughly with more regularity or more decay) the functional space, the easier it is to prove uniqueness (there are fewer solutions!) but the harder it is to prove existence (this regularity has to be shown on the solution!).
- Conversely the larger (i.e. less regular, weak solutions) the functional space, the easier it is to prove existence (of "weak" solutions, but it can remain still very difficult!), but the harder it is to prove uniqueness (there are lots of solutions!).
- One of the subtle point when studying a PDE or a class of PDE is to find the correct "balance" between these two requirements in the choice of the functional setting to achieve well-posedness.

Let us add finally as a summary that many notions, methods and questions studied in ODEs are *still highly useful* for PDEs, even if new phenomena arise, and one gain a lot of intuition by reflecting on the ODE theory:

- the question of solving the Cauchy problem *locally* with prescribed conditions,
- constructing solutions by fixed point arguments (Picard-Lindelöf),
- constructing solutions by approximation and compactness (Cauchy-Peano),
- the question of continuing the solution *globally* or not and the related *problem of blow-up in finite time* and *a priori estimates*,
- for certain equations, obtain formula for the solutions (separated variables, integral and Fourier representation…), "generalized" formula with entire series (Cauchy-Kovalevskaya)…
- linearization and perturbation problems (Hartman-Grobman…)

4. PDEs in science

4.1. PDEs and physics. The first revolution of ODEs with Leibniz and Newton, with the differential calculus but most importantly the first differential equation describing a physical law: the fundamental principle of dynamics of Newton. Combined with the universal law of gravitation, it led to the developement of mathematical celestial mechanics, and later to ballistic and so on… But, first of all, this was an immense conceptual revolution: expressing physical laws in terms of differential equations in order to predict the evolution of a system.

This novel idea then spread everywhere in science, and entered a second higher stage with Euler, Fourier and the birth of partial differential equations for modeling mathematically the evolution of continuum systems (fluids, heat conduction). Then PDEs accompanied each new field which emerged in modern science since then. Here are a few examples:

- The *compressible Euler equations*

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, \\ \partial_t E + \nabla \cdot (\mathbf{u}(E+p)) = 0, \quad E = \rho e + \frac{\rho |\mathbf{u}|^2}{2} \end{cases}$$

where e is the internal energy and p is given in terms of the unknowns $\rho, \mathbf{u}, E$ by a pressure law.

- The *incompressible Navier-Stokes equations* in fluid mechanics:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{\nabla p}{\rho} = \nu \Delta \mathbf{u}, \quad \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0, \quad \mathbf{u} = \mathbf{u}(t, x_1, x_2, x_3)$$

where ρ is the density of the fluid, $\mathbf{u}$ the velocity field of the fluid, E its energy, ν is the viscosity, and $\nabla := (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})^{\perp}$ and $\Delta := \nabla \cdot \nabla$.

- The *Maxwell equations* in electromagnetism (in the absence of source):

$$\begin{cases} \partial_t \mathbf{E} - \nabla \times \mathbf{B} = 0, \\ \partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \\ \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0. \end{cases}$$

with $\mathbf{E} = \mathbf{E}(t, x_1, x_2, x_3) \in \mathbb{R}^3$ and $\mathbf{B} = \mathbf{B}(t, x_1, x_2, x_3) \in \mathbb{R}^3$.

- The *Boltzmann equation* in kinetic theory:

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = Q(f,f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \left(f(\mathbf{v}') f(\mathbf{v}'_*) - f(\mathbf{v}) f(\mathbf{v}_*) \right) B(\mathbf{v} - \mathbf{v}_*, \sigma) \, \mathrm{d}\sigma \, \mathrm{d}\mathbf{v}_*$$

where $f = f(t, x_1, x_2, x_3, v_1, v_2, v_3) = f(t, \mathbf{x}, \mathbf{v}) \geq 0$ is integrable with total unit mass, B is the *collision kernel*, and

$$\mathbf{v}' = (\mathbf{v} + \mathbf{v}_*)/2 + |\mathbf{v} - \mathbf{v}_*| \sigma/2, \quad \mathbf{v}'_* = (\mathbf{v} + \mathbf{v}_*)/2 - |\mathbf{v} - \mathbf{v}_*| \sigma/2,$$

- The *Vlasov-Poisson equation* in kinetic theory:

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{F}[f] \cdot \nabla_{\mathbf{v}} f = 0, \quad \mathbf{F}[f] = \pm \nabla_{\mathbf{x}} \Delta_{\mathbf{x}}^{-1} \left(\rho[f] - \bar{\rho} \right)$$

where $\mathbf{F}$ is the gravitational or electric force field.

- The *Schrödinger equation* in quantum mechanics:

$$i \partial_t \psi = -\Delta_x \psi + V \psi$$

where $\psi = \psi(t, x_1, x_2, x_3) \in \mathbb{C}$ and V is a potential which can be given externally or depends on ψ (in the latter case this results in a nonlinear equation).

- The *Einstein equations* in general relativity

$$G_{ij} + g_{ij} \Lambda = 8 \pi T_{ij}$$

where g_{ij} is the metric tensor, $G_{ij} := R_{ij} - R g_{ij}/2$ is the Einstein tensor, R_{ij} the Ricci tensor, R the scalar curvature, T_{ij} is the stress-energy tensor, and Λ is the cosmological constant.

- Many other models for minimal surfaces, population/virus dynamics, coagulation-fragmentation... It is even used in financial mathematics with the Black and Scholes model, whose predicting power is quite debatable, but which certainly played a role in the dramatic expansion of derivative markets after the 70s!

4.2. PDEs in mathematics. However, it is often forgotten but differential equations also played and continue to play a major role in the developement of mathematics per se. It is rather obvious for ODEs as a specific area of mathematics has been developed out of the seminal works of Poincaré in order to understand qualitatively solutions to ODEs: dynamical systems. It is less obvious at first sight for PDEs but this a false impression. Let us give two examples.

The first example is that of complex variables. A complex function $f : \mathbb{C} \to \mathbb{C}$ that writes $f = f(x+iy) = u(x,y) + i\,v(x,y)$ is analytic if u and v are C^1 and satisfy **the Cauchy-Riemann equations** which are an example of partial differential equations:

$$\begin{cases} \partial_x u - \partial_y v = 0 \\ \partial_y u + \partial_x v = 0. \end{cases}$$

EXERCISE 9. *Show by using your course of complex variables that u and v are then C^2 and satisfy*

$$\partial^2_{xx} u + \partial^2_{yy} u = 0 \quad et \quad \partial^2_{xx} v + \partial^2_{yy} v = 0.$$

This leads to the notion of *harmonic functions*: $u : \mathbb{R}^\ell \to \mathbb{R}$ is harmonic if it is C^2 and its partial derivatives satisfy

$$\Delta u = \partial^2_{x_1 x_1} u + \cdots + \partial^2_{x_\ell x_\ell} u = 0.$$

These functions are studied in *harmonic analysis* and *potential theory*. They satisfy several fundamental properties: they are locally equal to their mean, and can be recovered from the expectation of a Brownian motion.

The second example concerns differential geometry. Many geometric problems about manifolds reframe as PDE evolution problems on the metric, curvature, etc. For instance the geometrization conjecture of W. Thurston (which includes the Poincaré conjecture) was reframed as a PDE problem by R. Hamilton, and then solved by G. Perelman in the early 2000s by constructing solutions to the Ricci flow

$$\partial_t g_{ij} = -2 R_{ij}$$

where g_{ij} is the metric tensor and R_{ij} is the Ricci tensor, and showing how to continue solutions beyond singularities by exploiting a new notion of entropy.

To conclude, let us emphasize that the study of PDEs has been at the core and many times the main motivation of the developement of many “pure” and “applied” areas in mathematics: Fourier-Laplace transforms and signal processing (exact solutions to PDEs), functional analysis in Banach and Hilbert spaces, integration theory, differential geometry, generalized functions (distributions, Sobolev spaces), calculus of variations, numerical analysis, harmonic analysis, microlocal analysis, etc.

5. Historical notes

5.1. On the question of "solving" equations. A long process has allowed for mathematicians to emancipate from exact formulas. The "first round" of this revolution succeeded, already in antiquity, to apply this point of view to the notion of ratios of magnitudes, or, in our modern language, real numbers. The first step was the understanding of rational numbers and how to calculate-write formulas for them, and then simple irrational numbers (actually square roots), with algorithm for calculating them (the Babylon-Newton algorithm...). The second conceptual leap was to realize that one can access certain real numbers by the methods of limits, and prove non-trivial relations about them, without being able to "solve" them, in the sense of a formula corresponding to a systematic and simple algorithm (such as for square roots). Example: The Eudoxian theory of incommensurable magnitudes and the relation – proven by Archimedes –that the surface area of a sphere of unit radius is twice the circumference of its equator. However at that stage historically the dominant idea by far remained that "to solve" means "to derive formulas".

This search for formulas was then systematically applied to the understanding of polynomial equations and algebraic numbers. Polynomials of order one and two had been understood since the middle-age, but a new impetus was given in Italy in the 16-17th centuries with the resolution "by radicals" (with formulas) of polynomials of degree 3 and 4. The hope for solving *all* polynomials this way were finally buried with Galois's theory in the early 19th century. With the new *differential equations* from Leibniz-Newton, the search for formulas started again, but the theory of Galois was extended to a class of simple differential equations (Riccati equations) by Picard-Vessot which showed again that formulas were hopeless in general.

The "second round" of this revolution, extending the limiting concept from numbers to functions that satisfy differential equations, is the central achievement of modern analysis. In this context, the first success of this method concerned ODEs. The first theorem of its type is the theorem (whose original form is in fact due to Cauchy) which states that solutions of the initial value problem to ODEs always exist for a maximal non-zero, possibly infinite, time T, complete with a characterization of what must happen if $T < \infty$ (in this form, the Picard-Lindelöf theorem). It was only in the late 19th century in the hands of Poincaré that the approach starting from this theorem became a method for "understanding" solutions. This gave rise to the so-called qualitative theory, born out of the observation that the programme of "solving the equations" as the primary tool for understanding them was dead.

It is remarkable that the step from one to several variables is so great that the subject acquires a completely different name – PDEs – and the requisite functional analysis on which everything must be based is of necessity much more rich. It is for this reason that a qualitative theory of PDEs had to wait until the twentieth century and it is this theory that these lectures concern.

This being said, one should not think that all the effort that went into writing explicit solutions has been useless. On the contrary, techniques which originally were motivated by "writing down the solution", e.g. Fourier analysis, fundamental solutions, etc., can now be conscripted to be used to understand fine properties of solutions one cannot write down. We shall see a little bit of this in these lectures.

5.2. History of PDEs. We refer to the nice historical paper [**2**] for the early history of PDEs, from which we extract some quotes in this paragraph. Leibniz used partial processes, but did not explicitly employ partial differential equations. In fact the conclusion is firmly established that neither Newton nor Leibniz in their published writings ever wrote down a partial differential equation and proceeded to solve it. Partial differential equations stand out clearly in six examples on trajectories published in 1719 by Nicolaus Bernoulli (1695-1726). Then partial differential equations are rarities in English articles of the eighteenth century and in English books (with the exception of Waring's). The latter was the only eighteenth century Englishman who wrote on partial differential equations other than the simplest types of the first order. The main contribution is Euler's Institutiones calculi integralis, Petropoli, 1770. Then Cauchy proves the local existence of analytic solutions to some class of PDEs in 1842 which is improved in its modern form by Sofia Kovalevskaya in 1875. The contribution of Hadamard Sur les problèmes aux dérivés partielles et leur signification physique in the Princeton bulletin in 1902 sets out the modern notion of well-posedness. The key contribution of Leray [**9**] on the incompressible Navier-Stokes equations is then one of the first major success in the nonlinear analysis of PDE, and marks the start of a new era.