CHAPTER 4

Hyperbolicity: scalar transport equations, wave equations

This chapter studies linear transport equations on some simple examples. The goal is to introduce rigorous classes of solutions for these PDEs thanks to the notion of characteristic trajectories. Weak solutions again shall be considered by duality formulation, and again we will not avoid presenting the complete theory of distributions.

The characteristic method is enlighting about the link between ODEs and PDEs. It reveals a certain of PDEs as a "continuum" of ODEs.

We shall in the last chapter about non-linearity some famous but simple problems about non-linear transport equations: the Vlasov equation, the Riemann problem for the Burgers equation.

Acknowledgements. This chapter is strongly inspired (in particular for weak and entropic solutions to the Burgers equation where we follow it almost exactly) by the excellent book of Denis Serre "*Systems of conservation laws 1: Hyperbolicity, Entropies, Shock Waves*", Cambridge University Press.

1. Introduction to the problem

"Transport phenomena" can mean particles –matter in general– transported along a flow (air, river, etc.), but it can also mean *information* transported, e.g. oscillations propagating for the wave equation. Applications of transport equations are vast and concern fluid mechanics, plasma physics, astrophysics, mathematical ecology, etc.

1.1. The class of equations. Consider a smooth open set $\mathcal{U} \subset \mathbb{R}^d$. Later in this chapter we will in fact mostly restrict ourself to the cases where $\mathcal{U} = \mathbb{R}^d$ is the whole space, or $\mathcal{U} = \mathbb{T}^d$ is the flat d-dimensional torus. There are the two simplest mathematical situation (unbounded and bounded) avoiding the presence of a boundary. In the presence of a boundary, the definition of the Cauchy problem should include boundary data, and discuss the question of whether the Cauchy hypersurface and Cauchy data are characteristic are not.

Let us make some recall about the main questions we want to ask about the *Cauchy problem*:

> Consider $T \in \mathbb{R}_+ \cup \{+\infty\}$ and $N \times N$ matrices $A_i(t, x, u)$, $1 \leq i \leq d$ that are smooth in $(t, x, u) \in [0, T] \times \mathcal{U} \times \mathbb{R}^N$. We search for $u(t, x)$ on $[0, T] \times \mathcal{U}$ and valued in $\mathbb{R}^N$, solution to

$$\partial_t u + \sum_{i=1}^{d} A_i(t, x, u)\, \partial_{x_i} u = 0, \quad (t, x) \in [0, T] \times \mathcal{U}. \tag{1.1}$$

REMARK 1.1. *Note that the dimension of the arrival space is in general different from that of the departure space, and they do not play the same role. The integer d is the number of space (or phase space for kinetic equations) coordinates, whereas N is the number of scalar unknowns and evolution equations. When $d = 1$ we say that the equation is* monodimensional, *and when $N = 1$ we say that the equation is* scalar.

We shall study the increasingly complicated following examples:

(1) The *monodimensional scalar equation* ($N = d = 1$)

$$\partial_t u + A(t, x, u)\, \partial_x u = 0. \tag{1.2}$$

Recall that even if $d = 1$, it is not an ODE because of the time variable ($\ell = 2$ in the previous chapters notation). This first case has already a rich structure and we shall look at first the linear equation with constant coefficient $A = c \in \mathbb{R}$, and then the linear case with space-dependent coefficient $A = A(x)$.

(2) Then we shall study the multidimensional ($d \geq 2$) scalar ($N = 1$) linear transport equation, with $A_i = A_i(t, x)$. This will illustrate the power of the characteristic methods. This class of equations includes the Liouville equation for particle systems.

(3) In the last chapter about nonlinear equations, we shall briefly study two nonlinear cases. First the monodimensional scalar equation with coefficient $A = A(u)$ depending on the solution but not time or space: $A(t, x, u) = f(u)$. This class of equations includes the Burgers equation and shows formation of shocks. Second we shall look at the Vlasov equation (for smooth interactions), which is a multidimensional scalar transport equation describing many mean-field many-body systems.

1.2. The initial value problem. Consider now $\mathcal{U} = \mathbb{R}^d$ (check as an exercise how to adapt everything in the case of $\mathcal{U} = \mathbb{T}^d$).

> **Problem 1 (existence).** Given u_0 on $\mathbb{R}^d$, is there at least one solution $u(t, x)$, $t \in [0, T]$, $x \in \mathbb{R}^d$, defined for a non-zero time $T > 0$, to the equation (1.1), with the initial data $u(0, x) = u_0(x)$, $x \in \mathbb{R}^d$?

As we saw previously to make this question precise, one needs to specify the functional space (or other restrictions) in which the solution is searched for, that is the *class* of solutions. When $T = +\infty$ we say that the solution is *global*. When the solution is regular enough for defining the derivatives appearing in the PDE in the classical differential calculus manner, we call it a *classical* or *strong* solution. Else the equation has to be reformulated by duality (or equivalently in the sense of distributions) and we call it a *weak* solution.

> **Problem 2 (uniqueness).** Given u_0 on $\mathbb{R}^d$ and $u_1(t, x)$ et $u_2(t, x)$ two solutions on $[0, T] \times \mathbb{R}^d$ with same initial data u_0, do we have $u_1 = u_2$?

Again to make this question precise, one has to specify the functional space. But now this is slightly more subtle than the existence problem. Not only one has to specify the functional space in which u_0, u_1, u_2 are considered, but also the functional space in which uniqueness is asked. And these two spaces can be different. For instance there can be uniqueness within the class of smooth solutions but infinitely many solutions in L^1, and moreover uniqueness of smooth solutions *within* L^1, which means if u_0 is smooth and u_1 is smooth and u_2 is L^1, then $u_1 = u_2$, implying in particular that u_2 is smooth. This is the basis of *weak-strong uniqueness principles.*

> **Problem 3 (regularity).** Given u_0 on $\mathbb{R}^d$ and, assuming existence, a solution $u(t, x)$ on $[0, T] \times \mathbb{R}^d$ with initial data u_0, where the initial data u_0 has a certain regularity (C^k or H^k or Hölder, etc.), then does the solution $u(t, x)$ enjoys this same regularity on the x-variable for $t \in [0, T]$?

Observe that the last question is often linked to the second one, as the regularity of the solution helps in proving uniqueness (think to Grönwall estimates). However for nonlinear transport equations, this is not true in general that the regularity is propagated, as shown for instance by the formation of shocks in fluid mechanics. The latter case also shows an interesting problem for uniqueness, where the L^∞ weak solutions are not unique, but one has to add some further *entropic conditions* guided by the intuition from physics in order to restore uniqueness.

1.3. Some examples.

1.3.1. *The linear transport.* Consider for $c \in \mathbb{R}$ ($N = d = 1$ here)

$$\begin{cases} \partial_t u + c\,\partial_x u = 0, & t \geq 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.3}$$

This models transport of particles on a line with algebraic velocity c, and u is the density of particles at time t and point x along the line.

1.3.2. *Burgers and traffic flow equations.* The *Burgers' equation* is the Euler in dimension 1 for the velocity field of a compressible gas:

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0. \tag{1.4}$$

The (simplest form of the) traffic flow equation is a nonlinear monodimensional scalar transport equation

$$\partial_t \rho + \partial_x \big(\rho\, u(\rho) \big) = 0. \tag{1.5}$$

Let us now briefly sketch the modelling of the latter equation. This is important to understand the meaning of models for guiding mathematical estimates and also this simple case allows to illustrate easily how to translate basic principles into PDEs.

Consider a road with only one line of cars, no entry or exit. Suppose that the typical size of a car is much smaller to the scale of the road and (more importantly) the scale at which we observe the road. We then model the car flow in a "continuum" manner, i.e. like a fluid. We call $\rho(t, x)$ the density of cars at t, x, $u(t, x)$ the velocity field of cars at t, x, and $q(t, x)$ their flux at t, x. (These quantities are in fact local averages on small portions of the road, e.g. one or two white bands…) Then the quantity of cars in $[x, x + \delta x]$ at time t is $\rho(t, x)\delta x$, and at time $t + \delta t$ is $\rho(t + \delta t, x)\delta x$. By conservation of the number of cars this is also the difference of the fluxes at x between times t and $t + \delta t$, minus the fluxes at $x + \delta x$ between times t and $t + \delta t$, which gives

$$\rho(t + \delta t, x)\delta x - \rho(t, x)\delta x = q(t, x)\delta t - q(t, x + \delta x)\delta t$$

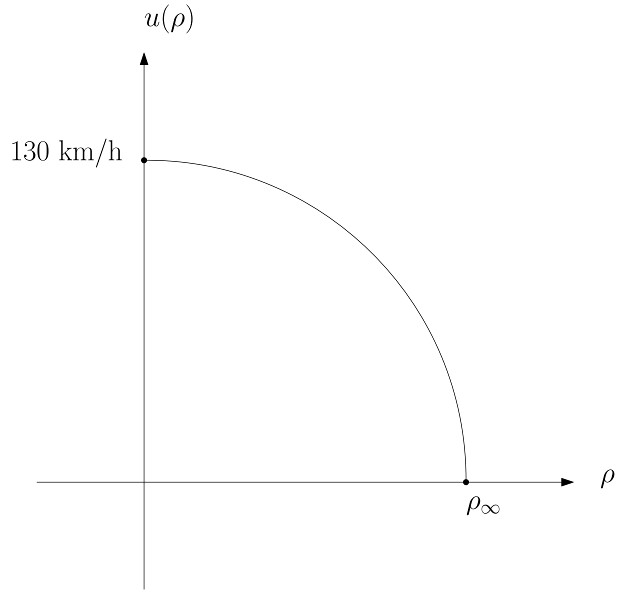

FIGURE 1.1. Drivers' behavior.

$$\text{and thus} \quad \frac{\rho(t+\delta t, x) - \rho(t,x)}{\delta t} + \frac{q(t, x+\delta x) - q(t,x)}{\delta x} = 0.$$

Taking $\delta t, \delta x \to 0$, we get $\partial_t \rho + \partial_x q = 0$. This equation is sometimes called the *continuity equation*, it expresses the conservation of the number of cars, and it belongs the class of so-called *conservation equations*, in hyperbolic equations.

This equation is not closed, it needs be complemented by a *state equation* allowing to determine the flux q in terms of the density ρ. In this case it will be deduced by observing the statistical behavior of drivers. Let us first write $q(t,x) = \rho(t,x)u(t,x)$ and assume that $u = u(\rho)$, i.e. saying that in first approximation the behavior of drivers is determined by density locally observed around them. The curve $\rho \mapsto u(\rho)$ is experiementally determined, in general decreasing from $u(0) = 130$ km/h down to $u(\rho_\infty) = 0$ with $\rho_\infty > 0$ the maximal density possible on this road.

We obtain finally the nonlinear transport PDE $\partial_t \rho + \partial_x \big(\rho\, u(\rho)\big) = 0$ as announced before. A typical theorem one can prove is that for the previous curve $u(\rho)$ there are in general developpement of traffic jams. The previous discussion is called a *derivation* of the equation, it is a formal (physics) discussion that can be sometimes transformed into a rigorous derivation when the solutions to some Cauchy problem of a more microscopical evolution can be shown to converge to the solutions to the Cauchy problem of the transport equation we derivate.

1.3.3. *Gas dynamics.* Euler equations for compressible gases in dimension 3 write

$$\partial_t \rho + \partial_{x_1}\big(\rho\, u_1\big) + \partial_{x_2}\big(\rho\, u_2\big) + \partial_{x_3}\big(\rho\, u_3\big) = 0 \tag{1.6}$$

$$\forall\, i = 1, \dots, d, \quad \partial_t(\rho\, u_i) + \sum_{j=1}^{d} \partial_j(u_j\, \rho\, u_i) + \partial_i p(\rho) = 0. \tag{1.7}$$

on the density ρ and velocity field u, complemented with a state equation on the pressure $p = p(\rho)$. We can write a third equation on the temperature field.

1.3.4. *The Liouville equation.* This is a transport equation on the density probability $F^n(t, X, V) = F^n(t, x_1, \ldots, x_n, v_1, \ldots, v_n)$ of n particles following Newton equations associated with the Hamilton function

$$H(X, V) = \sum_{i=1}^{n} \frac{|v_i|^2}{2} + \sum_{i \neq j} V(|x_i - x_j|) + \sum_{i=1}^{n} \phi(x_i)$$

with $x_i, v_i \in \mathbb{R}^m$. The equation writes in abstract form

$$\partial_t F^n + \nabla_V H \cdot \nabla_X F^n - \nabla_X H \cdot \nabla_V F^n = 0$$

or more explicitely

$$\partial_t F^n + V \cdot \nabla_X F^n - \sum_{i=1}^{n} \sum_{j \neq i} \nabla_{x_i} V(|x_i - x_j|) \cdot \nabla_{v_i} F^n - \nabla_X \phi \cdot \nabla_V F^n = 0.$$

This enters the previous framework as a linear scalar transport equation with $N = 1$ and $d = 2m$.

1.3.5. *The (Jeans)-Vlasov equation.* The Vlasov is obtained from the Liouville formally by the so-called mean-field limit. It writes (without external forces)

$$\partial_t f + v \cdot \nabla_x f = \int_{y,w} (\nabla_x \bar{V})(x - y) f(t, y, w) \nabla_v f(t, x, v) = \big(\nabla_x \bar{V} * \rho[f]\big) \cdot \nabla_v f$$

with

$$\rho[f](t, y) = \int_w f(t, y, w).$$

The function $\bar{V}$ is the interaction potential, and the force $\big(\nabla_x \bar{V} * \rho[f]\big)$ is self-induced by f. This is a multidimensional scalar nonlinear transport equation. When $\bar{V}$ is the Coulomb or Newton interaction potential, one recovers the Poisson equation $\Delta \mathcal{V} = \pm \rho$ on the mean-field potential $\mathcal{V} := \big(\bar{V} * \rho[f]\big)$, that we have studied before.

1.3.6. *The Wigner equation.* An interesting other case is that of the Wigner equation in quantum physics: consider the so-called Wigner distribution

$$w(t, x, \xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \rho\left(t, x + \frac{\eta}{2}\ ,\ x - \frac{\eta}{2}\right) e^{-i\xi\cdot\eta}\, \mathrm{d}\eta.$$

of the density matrix $\rho(t, x, y)$. Then it satisfies the Wigner nonlinear transport equation

$$\begin{cases} \partial_t w + \xi \cdot \nabla_x w + \Theta[V] w = 0, \\ w_{|t=0} = \ w_0(x, \xi), \end{cases}$$

with

$$(\Theta[V] f)(x, \xi) := -\frac{i}{(2\pi)^d} \iint_{\mathbb{R}^{2d}} \delta V(x, \eta)\, f(x, \xi')\ e^{i\eta\cdot(\xi - \xi')}\, \mathrm{d}\xi'\, \mathrm{d}\eta, \tag{1.8}$$

where the symbol δV is given by

$$\delta V(x, \eta) = V\left(x + \frac{\eta}{2}\right) - V\left(x - \frac{\eta}{2}\right). \tag{1.9}$$

For a pure state given by the wave function $\psi(t,x)$ the density matrix ρ is (the kernel of) the projection operator $|\psi\rangle\langle\psi|$, which satisfies the Schrödinger equation

$$i\hbar\partial_t\psi = H\psi, \quad H\psi = -\frac{\hbar}{2m}\Delta\psi + V\psi.$$

The density matrix satisfies the *Von Neumann equation*

$$i\hbar\partial_t\rho = [H,\rho]$$

where the RHS is given by a commutator when ρ is interpreted as an operator.

2. The linear transport with constant coefficient

Consider for $c\in\mathbb{R}$ and $T\in\mathbb{R}_+^*\cup\{+\infty\}$:

$$\begin{cases} \partial_t u + c\,\partial_x u = 0, \quad t\in[0,T],\ x\in\mathbb{R} \\ u(0,x) = u_0, \quad x\in\mathbb{R}. \end{cases} \tag{2.1}$$

2.1. Classical solutions.

Definition 2.1. *Assume that $u_0\in C^1(\mathbb{R})$. Then $u=u(t,x)$, $t\in[0,T]$, $x\in\mathbb{R}$ is called a* classical solution *to* (2.1) *with initial data u_0, if $u\in C^1([0,T]\times\mathbb{R})$ and u satisfies* (2.1) *in the sense of a pointwise equality between continuous functions.*

Theorem 2.2. *Assume that $u_0\in C^1(\mathbb{R})$. Then there exists a unique global classical solution to* (2.1)*. Moreover it is given by the following characteristic formula*

$$\forall\, t\ge 0,\ x\in\mathbb{R}, \quad u(t,x) = u_0(x-ct).$$

Remark 2.3. *Note that the uniqueness is true among all classical solutions (i.e. on any time interval* $[0,T]$*, not only the global ones. This is clear from the proof below.*

Proof. Let us consider separately the questions of existence and uniqueness.

Existence: Consider the function

$$u(t,x) := u_0(x-ct), \quad t\ge 0,\ x\in\mathbb{R}.$$

It belongs to $C^1(\mathbb{R}_+\times\mathbb{R}$ by composition rule, and satisfies $u(0,x)=u_0(x)$ for $x\in\mathbb{R}^d$. By chain rule, we compute its partial derivatives

$$\partial_t u = -c\,u_0'(x-ct), \quad \partial_x u = u_0'(x-ct)$$

and conclude that it satisfies the equation $\partial_t u + c\,\partial_x u = 0$ on $\mathbb{R}_+\times\mathbb{R}$, which shows existence.

Uniqueness: Consider a classical solution u on $[0,T]$ so that $u(0,x)=u_0$. Define

$$\varphi(t,x) = u(t,x+ct).$$

The function φ is C^1 by composition and

$$\partial_t\varphi = \partial_t u + c\,\partial_x u = 0.$$

We deduce by the fundamental theorem of calculus that

$$\forall\, t\in[0,T],\ x\in\mathbb{R}, \quad \varphi(t,x) = \varphi(0,x) = u_0(x)$$

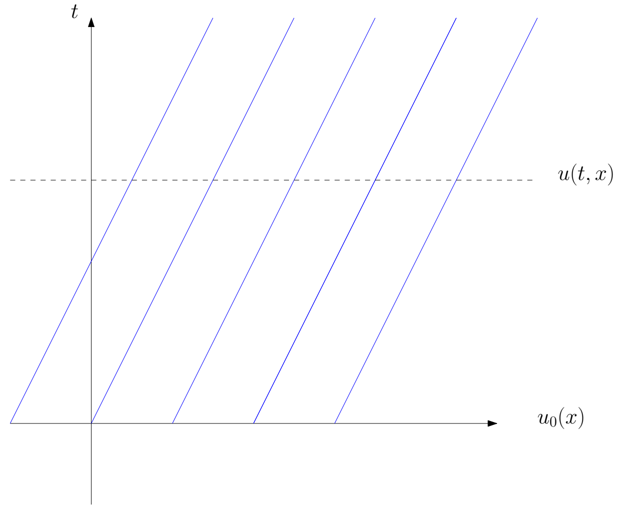

FIGURE 2.1. Picture of characteristics.

which proves that

$$\forall\, t \in [0,T],\ x \in \mathbb{R}, \quad u(t, x + ct) = u_0(x),$$

and shows that $u(t,x) = u_0(x - ct)$ is the unique solution constructed before. $\square$

Let us now add a source term to make the problem more complex. This shall be the occasion to introduce the Duhamel principle, which is the PDE version of the method of variation of the constant in ODE. Consider $h = h(t,x) \in C^1(\mathbb{R}_+ \times \mathbb{R})$ and the equation

$$\partial_t u + c\, \partial_x u = h, \quad \forall\, t \geq 0,\ x \in \mathbb{R}, \tag{2.2}$$

$$u(0,x) = u_0(x), \quad \forall\, x \in \mathbb{R}. \tag{2.3}$$

We define the notion of classical solutions as before.

DEFINITION 2.4. *Given $h \in C^1(\mathbb{R}_+ \times \mathbb{R})$ and $u_0 \in C^1(\mathbb{R})$, we call $u = u(t,x)$, $t \in [0,T]$, $x \in \mathbb{R}$ a classical solution to* (2.2)-(2.3) *for the initial data u_0 if $u \in C^1([0,T]\times\mathbb{R})$ and u satisfies* (2.2)-(2.3) *as a pointwise equality between continuous funtions.*

THEOREM 2.5. *Given $h \in C^1(\mathbb{R}_+ \times \mathbb{R})$ and $u_0 \in C^1(\mathbb{R})$, there exists a unique classical global solution to* (2.2)-(2.3)*. Moreover it is given by the characteristic formula*

$$u(t,x) = u_0(x - ct) + \int_0^t h(s, x - c(t-s))\, \mathrm{d}s.$$

PROOF. We proceed again in two steps, but beginning this time with the uniqueness part, as it has been treated already by the study of the homogeneous problem above.

Uniqueness: Consider u_1, u_2 two classical solutions to (2.2)-(2.3) on $[0,T]$ with initial data u_0. Then $v := u_1 - u_2$ is, by linearity, a classical solution to (2.1) with zero initial data. By Theorem 2.2, we deduce that $v = 0$ which concludes the proof.

Existence: One way is to start from the formula in the statement and check that it provides a classical solution: consider

$$u(t,x) = u_0(x-ct) + \int_0^t h(s, x - c(t-s))\,\mathrm{d}s.$$

This defines a $C^1(\mathbb{R}_+ \times \mathbb{R})$ function by composition and integration, which satisfies the initial condition $u(0,x) = u_0(x)$. The partial derivatives are

$$\partial_t u = -c\,u_0'(x-ct) + h(t,x) - c\int_0^t (\partial_x h)(s, x - c(t-s))\,\mathrm{d}s$$

$$\partial_x u = u_0'(x-ct) + \int_0^t (\partial_x h)(s, x - c(t-s))\,\mathrm{d}s.$$

which proves that $\partial_t u + c\,\partial_x u = h$ and concludes the proof of existence. □

Remark 2.6. *The previous explicit formula can be interpreted as the sum of the previous homogeneous formula (following backward the characteristic to find out the value of the initial data at $x - ct$) plus the action of the source term along this portion of the characteristic.*

Remark 2.7. *Let us now explain how to find the formula for the solution to* (2.2)-(2.3). *The idea is apply the method of variation of the constant along characteristics. In PDE this is callled Duhamel principle and it applies more generally along any semi-group. Let us argue by necessary condition. Consider a classical solution u to* (2.2)-(2.3) *and define $\varphi(t,y) = u(t, y+ct)$. Then this function φ is C^1 by composition and satisfies by chain-rule*

$$\partial_t \varphi(t,y) = (\partial_t u)(t, y+ct) + (\partial_x u)(t, x+ct) = h(t, y+ct).$$

Integrating along time, we get

$$\varphi(t,y) = \varphi(0,y) + \int_0^t h(s, y+cs)\,\mathrm{d}s$$

which means on u

$$u(t, y+ct) = u_0(y) + \int_0^t h(s, y+cs)\,\mathrm{d}s$$

and by changing variable $x = y + ct$:

$$u(t,x) = u_0(x-ct) + \int_0^t h(s, x + cs - ct)\,\mathrm{d}s.$$

Hence we have solved both the existence and uniqueness problems mentionned in the beginning of this chapter, in the case of C^1 solutions. Let us now give a unified statement (with or without source term, the case of no-source term being included in $f = 0$) solving the third problem of regularity.

Theorem 2.8. *Consider $h \in C^1(\mathbb{R}_+ \times \mathbb{R})$ and $u_0 \in C^1(\mathbb{R})$, and consider the classical C^1 solution u to* (2.2)-(2.3) *constructed above. Then if $u_0 \in C^k(\mathbb{R})$, $k \geq 1$ and $f \in C^k(\mathbb{R}_+ \times \mathbb{R})$, then this unique solution is $C^k(\mathbb{R}_+ \times \mathbb{R})$.*

Proof of Theorem 2.8. Exercise (use the explicit formulas.) □

2.2. Weak solutions. We introduce in this subsection notions of solutions *weaker* than the previous classical solutions, in the sense that they are not regular enough so that the derivatives appearing in the equation make sense for differential calculus. This has several motivations:

- A first motivation is the internal mathematical elegance and optimality of the theory: the formula we have established so far on the solution do make sense without assuming regularity on the solution; in fact they make sense for instance in L^∞.
- A second more motivation is that for nonlinear transport equation a C^1 regularity is often "too much to ask for" as this regularity can break down in finite time. We will see an example with the Burgers equations. This correspond to important physical phenomena (shock waves) that we want to describe and we have to develop a mathematical Cauchy theory that can capture them. Note that a discontinuity in a mathematical model does not correspond to a discontinuity in the real phenomenon, where there will be some small viscosity or other molecular effects that would be become non-negligible and if one "zoom in" on this "singularity"; a mathematical model is always an idealisation to a large extent, however it does not that the presence of these singularities in the solution is meaningless at all!
- A third motivation –that we will not have time to explore fully in this course– is that the kinetic Vlasov transport equation can be rigorously proved to be the limit of many-particle systems interacting through Newton laws, on the basis of a Cauchy theory for weak (measure) solutions.

Definition 2.9 (Weak L^∞ solutions). *Let $u_0 \in L^\infty(\mathbb{R})$. The function $u = u(t,x)$, $t \in \mathbb{R}_+$, $x \in \mathbb{R}$ is a weak L^∞ solution to* (2.1) *if: (i) $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$, and (ii)*

(2.4)
$$\forall \varphi \in C^1_c(\mathbb{R}_+ \times \mathbb{R}), \quad \int_{\mathbb{R}_+} \int_{\mathbb{R}} u(t,x)\ [\partial_t \varphi + c\,\partial_x \varphi]\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} u_0(x)\,\varphi(0,x)\,\mathrm{d}x = 0.$$

Remarks 2.10. *(1) Since $C^1_c(\mathbb{R}_+ \times \mathbb{R})$ contains in particular $\mathcal{D}(\mathbb{R}^*_+ \times \mathbb{R})$, the property* (7.1) *implies the equation* $\partial_t u + c\,\partial_x u = 0$ in the sense of distributions, *i.e. in the space $\mathcal{D}'(\mathbb{R}^*_+ \times \mathbb{R})$. Be careful when you write this equation: this is an equality between linear forms, not between functions since in general the derivatives do not exist as functions (take for instance for u the heaviside function on $\mathbb{R}$). We will also say that u "satisfies the equation in the dual or weak sense".*

(2) Note that one could define the notion of weak solutions in spaces larger than L^∞: measure solutions, or even distribution solutions. Note also that the initial data requires some minimal regularity on the time variable to make sense, which does not seem to be included in the assumption $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$. It turns out that the weak formulation (7.1) *itself implies some continuity in time (valued in a weak space like $W^{-1,\infty}(\mathbb{R})$ or $\mathcal{D}'(\mathbb{R})$) by relating the first order time derivative to the other derivatives.*

(3) This definition could be extended to any time interval $[0,T]$*, just like all this section. We do not write it for the sake of simplicity.*

EXERCISE 46. *Prove that the property* (7.1) *is equivalent to the same property where one replaces* $\varphi \in C^1_c(\mathbb{R}_+ \times \mathbb{R})$ *by* $\varphi \in C^\infty_c(\mathbb{R}_+ \times \mathbb{R}) = \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$.

This first thing to check when one defines weak solutions is that they really *extend* the already defined classical solutions, i.e. if classical and weak solutions do exist do they coincide, and if weak solutions are regular are they classical?

THEOREM 2.11. *Let* $u_0 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. *First if* u *is a classical solution to* (2.1) *then* u *is also a weak solution to* (2.1). *Second, if* u *is a weak solution to* (2.1) *and* $u \in C^1(\mathbb{R}_+ \times \mathbb{R})$, *then* u *is a classical solution.*

PROOF OF THEOREM 2.11. First let us consider a classical solution u and $\varphi \in C^1(\mathbb{R}_+ \times \mathbb{R})$. Then from the explicit formula $u(t,x) = u_0(x - ct)$, we have $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$, and from the PDE we get

$$\varphi\left(\partial_t u + c\,\partial_x u\right) = 0.$$

Integrating on $\mathbb{R}_+ \times \mathbb{R}$, we deduce by integration by parts

$$-\int_{\mathbb{R}_+}\int_{\mathbb{R}} u\big(\partial_t\varphi + c\,\partial_x\varphi\big)\,\mathrm{d}t\,\mathrm{d}x - \int_{\mathbb{R}} u(0,x)\varphi(0,x)\,\mathrm{d}x = 0$$

which shows (7.1) using that $u(0,x) = u_0(x)$, and proves that u is a weak solution.

Second assume that u is a weak solution with initial data u_0, and with $u \in C^1(\mathbb{R}_+ \times \mathbb{R})$. We first consider the weak formulation (7.1) with any test function $\varphi \in C^1_c(\mathbb{R}^*_+ \times \mathbb{R})$ (support avoiding the initial time) and compute by integration by parts (since u is C^1)

$$0 = \int_{\mathbb{R}_+}\int_{\mathbb{R}} u\big(\partial_t\varphi + c\,\partial_x\varphi\big)\,\mathrm{d}t\,\mathrm{d}x = -\int_{\mathbb{R}_+}\int_{\mathbb{R}} \big(\partial_t u + c\,\partial_x u\big)\varphi\,\mathrm{d}t\,\mathrm{d}x$$

which implies that $\partial_t u + c\partial_x u = 0$ on $\mathbb{R}^*_+ \times \mathbb{R}$. Finally we consider any test function $\psi \in C^1_c(\mathbb{R})$ and then build a test function $\varphi \in C^1_c(\mathbb{R}_+ \times \mathbb{R})$ so that $\varphi(0,\cdot) = \psi$. We write

$$\begin{aligned}
0 &= \int_{\mathbb{R}_+}\int_{\mathbb{R}} u\big(\partial_t\varphi + c\,\partial_x\varphi\big)\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} u_0(x)\psi(x)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}_+}\int_{\mathbb{R}} \big(\partial_t u + c\,\partial_x u\big)\varphi\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} \left(u_0(x) - u(0,x)\right)\psi(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \left(u_0(x) - u(0,x)\right)\psi(x)\,\mathrm{d}x
\end{aligned}$$

and we deduce that $u_0 = u(0,\cdot)$ on $\mathbb{R}$, which concludes the proof that u is a classical C^1 solution. $\square$

REMARK 2.12. *To sum up, [strong] implies [weak], and [weak + regularity] implies [strong].*

THEOREM 2.13. *Let* $u_0 \in L^\infty$, *then the Cauchy problem* (2.1) *admits a unique global* L^∞ *weak solution. This solution also satisfies* $u(t,x) = u_0(x - ct)$ *almost everywhere in* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

PROOF. We proceed in two steps, distinguishing existence and uniqueness again.

Existence: The formula $u(t,x) = u_0(x-ct)$ defines an L^∞ function. Then we consider $\varphi \in C^1_c(\mathbb{R}_+ \times \mathbb{R})$ and compute

$$I := \int_{\mathbb{R}_+} \int_{\mathbb{R}} u(t,x)\big(\partial_t \varphi + c\,\partial_x \varphi\big)\,\mathrm{d}t\,\mathrm{d}x = \int_{\mathbb{R}_+} \int_{\mathbb{R}} u_0(x-ct)\big(\partial_t \varphi + c\,\partial_x \varphi\big)\,\mathrm{d}t\,\mathrm{d}x$$

which writes with the change of variable $X = x - ct$:

$$I = \int_{t \in \mathbb{R}_+} \int_{X \in \mathbb{R}} u_0(X)\big(\partial_t \varphi(t, X+ct) + c\,\partial_x \varphi(t, X+ct)\big)\,\mathrm{d}t\,\mathrm{d}X$$

and we use the chain rule

$$\partial_t\big[\varphi(t, X+ct)\big] = \partial_t \varphi(t, X+ct) + c\,\partial_x \varphi(t, X+ct)$$

which yields

$$I = \int_{t \in \mathbb{R}_+} \int_{X \in \mathbb{R}} u_0(X)\,\partial_t\big[\varphi(t, X+ct)\big]\,\mathrm{d}t\,\mathrm{d}X.$$

We finally perform an integration by parts in t (keeping X fixed)

$$I = -\int_{X \in \mathbb{R}} u_0(X)\,\varphi(0, X)\,\mathrm{d}X$$

which proves (7.1) et concludes the proof of existence.

Uniqueness: Consider two weak L^∞ solutions u_1 and u_2 with the same initial data $u_0 \in L^\infty(\mathbb{R})$, and their difference $v := (u_1 - u_2)$. The equation and its weak formulation being linear, v is again a weak solution of (2.1), with zero initial data. We want to prove $v = 0$ in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$, (i.e. almost everywhere on $\mathbb{R}_+ \times \mathbb{R}$). It is hence enough to prove

$$\forall \psi \in C^1_c(\mathbb{R}_+ \times \mathbb{R}), \quad \int_{\mathbb{R}_+} \int_{\mathbb{R}} v(t,x)\,\psi(t,x)\,\mathrm{d}t\,\mathrm{d}x = 0. \tag{2.5}$$

We first claim that to prove (3.9) it is enough to prove

For any $\psi \in C^1_c(\mathbb{R}_+ \times \mathbb{R})$, there is $\varphi \in C^1_c(\mathbb{R}_+ \times \mathbb{R})$ so that

$$\partial_t \varphi + c\,\partial_x \varphi = \psi, \quad t \geq 0,\ x \in \mathbb{R}. \tag{2.6}$$

Indeed, since v is a weak solution with zero initial data we have

$$\forall\,\varphi \in C^1_c(\mathbb{R}_+ \times \mathbb{R}), \quad \int_{\mathbb{R}_+} \int_{\mathbb{R}} v(t,x)\,\big[\partial_t \varphi + c\,\partial_x \varphi\big]\,\mathrm{d}t\,\mathrm{d}x = 0$$

which implies (3.9) as soon as we have (3.10).

Let us now prove (3.10). We have solved in Theorem 2.5 the linear problem with source term: there is a unique C^1 solution given by

$$\varphi(t,x) = \varphi_0(x-ct) + \int_0^t \psi(s, x - c(t-s))\,\mathrm{d}s.$$

What remains to be proved in order to solve (3.10) is that, by playing with the different initial data φ_0 we can build a compactly supported solution. Let us define

$$\forall\, x \in \mathbb{R}, \quad \varphi_0(x) := -\int_0^T \psi(s, x + cs)\,\mathrm{d}s. \tag{2.7}$$

This is a C^1 compactly supported initial data, and the associated solution to (3.9) is

$$\begin{aligned}
\varphi(t,x) =& \varphi_0(x-ct) + \int_0^t \psi(s, x - c(t-s))\,\mathrm{d}s \\
=& -\int_0^T \psi(s, x - ct + cs)\,ds + \int_0^t \psi(s, x - c(t-s))\,\mathrm{d}s \\
=& \int_T^t \psi(s, x - c(t-s))\,\mathrm{d}s.
\end{aligned}$$

We know that ψ has compact support, say included in $[0,T]\times[-R,R]$. Then for $t \geq T$, the function φ vanishes since the integrand above vanishes. Second, for $t \in [0,T]$, one has $|t-s| \in [0,T]$ since $0 \leq t \leq s \leq T$, and therefore $\varphi(t,x)$ vanishes as soon as $|x - c(t-s)| \geq R$ for all s, which is true as soon as $|x| \geq R + cT$. Finally we deduce that φ has compact support in $[0,T]\times[-R-cT, R+cT]$. This concludes the proof. $\square$

Exercise 47. *Generalise the previous results to the case $x \in \mathbb{R}^d$, $c \in \mathbb{R}^d$ and*

$$\partial_t u(t,x) + c \cdot \nabla_x u(t,x) = 0.$$

This corresponds to different propagation speeds along the different coordinates of the problem.

3. The linear scalar transport equation with variable coefficients

3.1. The setting. Consider the following equation

$$\left\{\begin{array}{l} \partial_t u(t,x) + \mathbf{F}(t,x)\cdot\nabla_x u(t,x) = 0, \qquad t \in \mathbb{R},\ x \in \mathbb{R}^d \\ u(t=0,x) = u_0(x), \qquad x \in \mathbb{R}^d. \end{array}\right. \tag{3.1}$$

The propagation speeds $\mathbf{F}(t,x) \in \mathbb{R}^d$ can now depend on time and space and be different in each coordinate.

We assume that $u_0 \in C^1(\mathbb{R}^d)$, and we assume on the *vector field* $\mathbf{F}$ that (a) $\mathbf{F} \in C^1(\mathbb{R}\times\mathbb{R}^d)$, and

$$(b) \qquad \forall\, t \geq 0,\ x \in \mathbb{R}, \quad |\nabla_x \mathbf{F}(t,x)| \leq L \tag{3.2}$$

for some constant $L > 0$.

Remark 3.1. *In the sequel, one can replace (exercise) the assumption (b) with*

$$(b') \qquad \forall\, t \geq 0,\ x \in \mathbb{R}, \quad \mathbf{F}(t,x) \leq L\,(1 + |x|).$$

Let us now explain how to build solutions to this PDE thanks to the ODE theory and the *method of characteristics.*

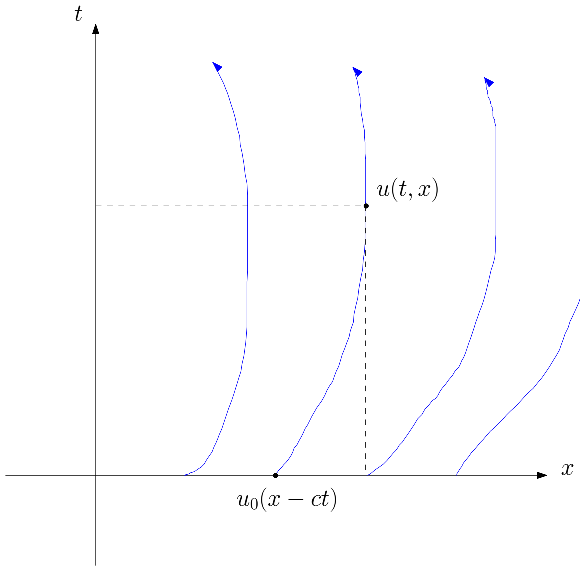

FIGURE 3.1. Picture of characteristics.

DEFINITION 3.2. *We call* characteristics of the transport equation (3.1)*, the trajectories* $(Z_{s,t})$*,* $s,t \in R$*, of the differential system*

$$\begin{cases} \partial_t Z_{s,t}(x) = \mathbf{F}(t, Z(t,x)), & t \geq 0, \\ Z_{s,s}(x) = x \in \mathbb{R}^d. \end{cases}$$

We have from the ODE theory the following theorem

THEOREM 3.3. *Assuming that* $\mathbf{F}$ *satisfies (a)-(b), then these trajectories exist for any* $s,t \geq 0$*, and moreover for any* $s,t \geq 0$*, the map*

$$Z_{s,t} : \mathbb{R}^d \to \mathbb{R}^d, \quad x \mapsto Z_{s,t}(x)$$

is a C^1*-diffeomorphism (which means for instance that the trajectories do not cross).*

In general there is no semigroup structure but one has always

$$\forall\, t_0, t_1, t_2 \geq 0, \quad Z_{t_1,t_2} \circ Z_{t_0,t_1} = Z_{t_0,t_2}.$$

PROOF. The proof follows from the Picard-Lindelöf theorem. The construction of global trajectories $Z_{s,t}(x)$ is directly provided by the theorem, and then the fact that $Z_{s,t}$ is C^1 comes from the C^1 dependency according to the initial data in Picard-Lindelöf theorem, and finally the fact that it is a C^1-diffeomorphism comes from the fact that $Z_{t,s}$ exists, is also C^1, and satisfies $Z_{t,s} = Z_{s,t}^{-1}$. □

REMARK 3.4. *Be careful that here there is only one variable* x *and the differential system has order* 1*. In case of kinetic transport equations, the associated characteristic differential system has order two and two variables* x, v*. In this case the fact that the trajectories do not cross should be understood properly as "do not cross in the phase space* (x,v)*".*

The heuristic in the picture suggests that $u(t,x) = u_0(Z_{t,s}(x))$, this is the object of the next subsections.

3.2. Well-posedness for classical solutions.

DEFINITION 3.5. *Let $u_0 \in C^1(\mathbb{R}^d)$ and $\mathbf{F} \in C^1(\mathbb{R} \times \mathbb{R}^d)$ satisfying (a)-(b). A classical solution to* (3.1) *is a function $u \in C^1(\mathbb{R} \times \mathbb{R}^d; \mathbb{R})$ which satisfies* (3.1) *in the sense of an equality between continuous functions in $\mathbb{R} \times \mathbb{R}^d$.*

THEOREM 3.6. *Let $u_0 \in C^1(\mathbb{R}^d)$ and $\mathbf{F}$ satisfies (a)-(b). Then there is a unique global classical solution $u \in C^1(\mathbb{R} \times \mathbb{R}^d)$ to* (3.1). *Moreover it is given by the characteristic method either in implicit form*

$$\forall\, t \in \mathbb{R},\ \forall\, x \in \mathbb{R}^d, \quad u\left(t, Z_{0,t}(x)\right) = u_0(x). \tag{3.3}$$

or in explicit form

$$\forall\, t \in \mathbb{R},\ \forall\, x \in \mathbb{R}^d, \quad u(t,x) = u_0\left(Z_{t,0}(x)\right). \tag{3.4}$$

REMARK 3.7. *Observe on the picture that the solution is "constant along the characteristic trajectories". This characteristic method is hence of great importance both in applications for solving many scalar transport PDEs, but also conceptually as it gives a concrete bridge between the Cauchy theories for ODEs and PDEs, and shows in some cases how to interpret a PDE as a continuum of ODEs.*

PROOF OF THEOREM 3.9. We proceed in two steps as before.

Uniqueness: Assume that $u \in C^1$ is a solution to (3.1) with initial data u_0. From the chain rule we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[u\left(t, Z_{0,t}(x)\right)\right] &= \left(\partial_t u\right)\left(t, Z_{0,t}(x)\right) + \left(\nabla_x u\right)\left(t, Z_{0,t}(x)\right) \cdot \partial_t Z_{0,t}(x) \\ &= \left(\partial_t u + \mathbf{F} \cdot \nabla_x u\right)\left(t, Z_{0,t}(x)\right) = 0,\end{aligned}$$

which shows that

$$u\left(t, Z_{0,t}(x)\right) = u\left(0, Z_{0,0}(x)\right) = u_0\left(x\right)$$

since $Z_{0,0}(x) = x$, and thus u satisfies (3.3). This proves uniqueness since it characteristizes the solution and also shows (3.4) since $Z_{0,t}$ is a C^1-diffeomorphism for any $t \in \mathbb{R}$.

Existence: Consider the application

$$\forall\, x \in \mathbb{R}^d, \quad w(t,x) := u_0\left(Z_{t,0}(x)\right)$$

which is C^1 in both variables by composition, and satisfies the initial condition since $Z_{0,0} = \mathrm{Id}$:

$$\forall\, x \in \mathbb{R}^d, \quad w(0,x) = u_0\left(Z_{0,0}(x)\right) = u_0(x).$$

It also satisfies

$$\forall\, t \in \mathbb{R},\ x \in \mathbb{R}^d, \quad w\left(t, Z_{0,t}(x)\right) = u_0\left(Z_{t,0} \circ Z_{0,t}(x)\right) = u_0(x).$$

By differentiating in time the last equation one gets (following the same chain rule calculation as above):

$$\forall\, t \in \mathbb{R},\ x \in \mathbb{R}^d, \quad \left(\partial_t w + F \cdot \nabla w\right)\left(t, Z_{0,t}(x)\right) = 0.$$

Defining $x = Z_{t,0}(y)$ for any $y \in \mathbb{R}^d$ as $t \in \mathbb{R}$ is fixed we deduce

$$\forall\, t \in \mathbb{R},\ y \in \mathbb{R}^d, \quad (\partial_t w + \mathbf{F} \cdot \nabla w)\,(t, y) = 0$$

which concludes the proof. $\square$

Remark 3.8. *Another proof of existence would start directly from the explicit formula to check that w satisfies the PDE. It would lead to more complicated calculations. Let us sketch briefly how it goes: from*

$$\forall\, x \in \mathbb{R}^d, \quad Z_{0,t}(Z_{t,0}(x)) = x$$

one gets by differentiating in t and x

$$\begin{cases} (\partial_t Z_{0,t})\,(Z_{t,0}(x)) + \partial_t\,(Z_{t,0}(x)) \times (J_x Z_{0,t})\,(Z_{t,0}(x)) = 0, \\ J_x\,(Z_{t,0}(x)) \times (J_x Z_{0,t})\,(Z_{t,0}(x)) = \mathrm{Id}. \end{cases}$$

Then define w as $w(t,x) := u_0\left(Z_t^{-1}(x)\right)$ and multiply the desired equation on w by the invertible matrix $(J_x Z_{0,t})(Z_{t,0}(x))$:

$$\begin{aligned} &[\partial_t w(t,x) + \mathbf{F}(t,x) \cdot \nabla_x w(t,x)] \times (J_x Z_{0,t})\,(Z_{t,0}(x)) \\ &= \nabla_x u_0\,(Z_{t,0}(x)) \cdot [\partial_t\,(Z_{t,0}(x)) + \mathbf{F}(t,x) \cdot J_x\,(Z_{t,0}(x))] \times (J_x Z_{0,t})\,(Z_{t,0}(x)) \\ &= \nabla_x u_0\,(Z_{t,0}(x)) \cdot [\mathbf{F} t, x) - (\partial_t Z_{0,t})\,(t, Z_{t,0}(x))]\,. \end{aligned}$$

Finally use that from the definition of the characteristic trajectories

$$(\partial_t Z_{0,t})\,(t, Z_{t,0}(x)) = \mathbf{F}\,(t, Z_{0,t} \circ Z_{t,0}(x)) = \mathbf{F}(t,x)$$

which implies that

$$[\partial_t w(t,x) + \mathbf{F}(t,x) \cdot \nabla_x w(t,x)] \times (J_x Z_{0,t})\,(t, Z_{t,0}(x)) = 0$$

and thus

$$\partial_t w(t,x) + \mathbf{F}(t,x) \cdot \nabla_x w(t,x) = 0.$$

Let us now extend our result on classical solutions to the case of a source term. Consider the equation

$$\begin{cases} \partial_t u(t,x) + \mathbf{F}(t,x) \cdot \nabla_x u(t,x) = h(t,x), & t \in \mathbb{R},\ x \in \mathbb{R}^d \\ u(t=0,x) = u_0(x), & x \in \mathbb{R}^d \end{cases} \tag{3.5}$$

with $h \in C^1(\mathbb{R}_+ \times \mathbb{R})$.

Theorem 3.9. *Let $u_0 \in C^1(\mathbb{R}^d)$ and $\mathbf{F}, h \in C^1(\mathbb{R} \times \mathbb{R}^d)$ with $\mathbf{F}$ satisfying (a)-(b). Then there is a unique global solution $u \in C^1(\mathbb{R} \times \mathbb{R}^d)$ to the problem* (3.5) *and it is given by*

$$\forall\, t \in \mathbb{R},\ \forall\, x \in \mathbb{R}^d, \quad u\,(t, Z_{0,t}(x)) = u_0(x) + \int_0^t h(s, Z_{0,s}(x))\,\mathrm{d}s. \tag{3.6}$$

or in explicit form

$$\forall\, t \in \mathbb{R},\ \forall\, x \in \mathbb{R}^d, \quad u(t,x) = u_0\,(Z_{t,0}(x)) + \int_0^t h(s, Z_{t,s}(x))\,\mathrm{d}s. \tag{3.7}$$

PROOF. The proof is based on the calculation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[u\left(t, Z_{0,t}(x)\right)\right] = h(t, Z_{0,t}(x))$$

when u satisfies the equation. The rest of the proof is similar to the case of constant coefficients. $\square$

3.3. Well-posedness for weak solutions. We restrict in this subsection to the case of a *divergence free* vector field F, i.e. we make the further assumption

$$(c) \qquad \forall\, t \geq 0,\ x \in \mathbb{R}^d, \quad \nabla_x \cdot \mathbf{F}(t,x) = 0.$$

DEFINITION 3.10. *Let $u_0 \in L^\infty(\mathbb{R}^d)$ and $\mathbf{F} \in C^1(\mathbb{R} \times \mathbb{R}^d)$ satisfying (a)-(b). A weak L^∞ solution to* (3.1) *is a function $u \in L^\infty(\mathbb{R} \times \mathbb{R}^d; \mathbb{R})$ which satisfies the weak formulation*

$$\forall\, \varphi \in C^1_c(\mathbb{R}_+ \times \mathbb{R}), \quad \int_{\mathbb{R}_+}\int_{\mathbb{R}^d} u(t,x)\ [\partial_t \varphi + \mathbf{F} \cdot \nabla_x \varphi]\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}^d} u_0(x)\,\varphi(0,x)\,\mathrm{d}x = 0. \tag{3.8}$$

As before we first check the consistence with previous classical solutions:

THEOREM 3.11. *Let $u_0 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\mathbf{F}$ satisfies (a)-(b)-(c). First if u is a classical solution to* (3.1) *then u is also a weak solution to* (3.1)*. Second, if u is a weak solution to* (3.1) *and $u \in C^1(\mathbb{R}_+ \times \mathbb{R})$, then u is a classical solution to* (3.1)*.*

PROOF OF THEOREM 3.11. First let us consider a classical solution u and $\varphi \in C^1(\mathbb{R}_+ \times \mathbb{R})$. Then from the explicit formula $u(t,x) = u_0(Z_{t,0}(x))$, we have $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$, and from the PDE we get

$$\varphi\left(\partial_t u + \mathbf{F} \cdot \nabla_x u\right) = 0.$$

Integrating on $\mathbb{R}_+ \times \mathbb{R}^d$, we deduce by integration by parts

$$-\int_{\mathbb{R}_+}\int_{\mathbb{R}^d} u\big(\partial_t \varphi + \mathbf{F} \cdot \nabla_x \varphi\big)\,\mathrm{d}t\,\mathrm{d}x - \int_{\mathbb{R}_+}\int_{\mathbb{R}^d} u\varphi(\nabla_x \cdot \mathbf{F})\,\mathrm{d}t\,\mathrm{d}x - \int_{\mathbb{R}^d} u(0,x)\varphi(0,x)\,\mathrm{d}x = 0$$

which shows (3.8) using that $u(0,x) = u_0(x)$ and $\nabla_x \cdot \mathbf{F} = 0$, and proves that u is a weak solution.

Second assume that u is a weak solution with initial data u_0, and with $u \in C^1(\mathbb{R}_+ \times \mathbb{R})$. We first consider the weak formulation (3.8) with any test function $\varphi \in C^1_c(\mathbb{R}^*_+ \times \mathbb{R})$ (support avoiding the initial time) and compute by integration by parts (since u is C^1)

$$0 = \int_{\mathbb{R}_+}\int_{\mathbb{R}^d} u\big(\partial_t \varphi + \mathbf{F} \cdot \nabla_x \varphi\big)\,\mathrm{d}t\,\mathrm{d}x = -\int_{\mathbb{R}_+}\int_{\mathbb{R}^d} \big(\partial_t u + \mathbf{F} \cdot \nabla_x u\big)\varphi\,\mathrm{d}t\,\mathrm{d}x$$

using again $\nabla_x \cdot F = 0$, which implies that $\partial_t u + \mathbf{F} \cdot \nabla_x u = 0$ on $\mathbb{R}^*_+ \times \mathbb{R}$. Finally we consider any test function $\psi \in C^1_c(\mathbb{R})$ and then build a test function $\varphi \in C^1_c(\mathbb{R}_+ \times \mathbb{R})$ so that $\varphi(0,\cdot) = \psi$. We write

$$\begin{aligned} 0 &= \int_{\mathbb{R}_+}\int_{\mathbb{R}^d} u\big(\partial_t \varphi + \mathbf{F} \cdot \nabla_x \varphi\big)\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}^d} u_0(x)\psi(x)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}_+}\int_{\mathbb{R}^d} \big(\partial_t u + \mathbf{F} \cdot \nabla_x u\big)\varphi\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}^d} \left(u_0(x) - u(0,x)\right)\psi(x)\,\mathrm{d}x \end{aligned}$$

$$= \int_{\mathbb{R}^d} (u_0(x) - u(0,x))\, \psi(x)\, \mathrm{d}x$$

and we deduce that $u_0 = u(0,\cdot)$ on $\mathbb{R}^d$, which concludes the proof that u is a classical C^1 solution. $\square$

THEOREM 3.12. *Let $u_0 \in L^\infty(\mathbb{R}^d)$ and $\mathbf{F}$ satisfies (a)-(b)-(c). Then there is a unique global weak solution $u \in L^\infty(\mathbb{R}\times\mathbb{R}^d)$ to* (3.1). *Moreover it is given again by the characteristic method either in implicit form* (3.3) *or in explicit form* (3.4).

PROOF OF THEOREM 3.12. We proceed in two steps.

Existence: The formula $u(t,x) = u_0(Z_{t,0}(x))$ defines an L^∞ function. Then we consider $\varphi \in C^1_c(\mathbb{R}_+ \times \mathbb{R})$ and compute

$$I := \int_{\mathbb{R}_+}\int_{\mathbb{R}^d} u(t,x)\big(\partial_t\varphi + \mathbf{F}\cdot\nabla_x\varphi\big)\,\mathrm{d}t\,\mathrm{d}x = \int_{\mathbb{R}_+}\int_{\mathbb{R}} u_0(Z_{t,0}(x))\big(\partial_t\varphi + \mathbf{F}\cdot\nabla_x\varphi\big)\,\mathrm{d}t\,\mathrm{d}x$$

which writes with the change of variable $X = Z_{t,0}(x)$:

$$I = \int_{\mathbb{R}_+}\int_{\mathbb{R}} u_0(X)\big(\partial_t\varphi(t,Z_{0,t}(X)) + \mathbf{F}(t,Z_{0,t}(X))\cdot\partial_x\varphi(t,Z_{0,t}(X))\big)\,\mathrm{d}t\,\mathrm{d}X$$

and we use the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\varphi(t,Z_{0,t}(X))\big] = \partial_t\varphi(t,Z_{0,t}(X)) + \mathbf{F}\cdot\nabla_x\varphi(t,Z_{0,t}(X))$$

which yields

$$I = \int_{\mathbb{R}_+}\int_{\mathbb{R}} u_0(X)\frac{\mathrm{d}}{\mathrm{d}t}\big[\varphi(t,Z_{0,t}(X))\big]\,\mathrm{d}t\,\mathrm{d}X.$$

We finally perform an integration by parts in t (keeping X fixed)

$$I = -\int_{X\in\mathbb{R}} u_0(X)\,\varphi(0,X)\,\mathrm{d}X$$

which proves (7.1) et concludes the proof of existence.

Uniqueness: Consider two weak L^∞ solutions u_1 and u_2 with the same initial data $u_0 \in L^\infty(\mathbb{R})$, and their difference $v := (u_1 - u_2)$. The equation and its weak formulation being linear, v is again a weak solution of (2.1), with zero initial data. We want to prove $v = 0$ in $L^\infty(\mathbb{R}_+\times\mathbb{R}^d)$, (i.e. almost everywhere on $\mathbb{R}_+\times\mathbb{R}^d$). It is hence enough to prove

$$\forall\,\psi \in C^1_c(\mathbb{R}_+\times\mathbb{R}^d), \quad \int_{\mathbb{R}_+}\int_{\mathbb{R}^d} v(t,x)\,\psi(t,x)\,\mathrm{d}t\,\mathrm{d}x = 0. \tag{3.9}$$

We first claim that to prove (3.9) it is enough to prove

For any $\psi \in C^1_c(\mathbb{R}_+\times\mathbb{R}^d)$, there is $\varphi \in C^1_c(\mathbb{R}_+\times\mathbb{R}^d)$ so that

$$\partial_t\varphi + \mathbf{F}\cdot\nabla_x\varphi = \psi, \quad t \ge 0,\ x\in\mathbb{R}^d. \tag{3.10}$$

Indeed, since v is a weak solution with zero initial data we have

$$\forall \varphi \in C^1_c(\mathbb{R}_+ \times \mathbb{R}^d), \quad \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} v(t,x) \left[\partial_t \varphi + c\, \partial_x \varphi\right] \mathrm{d}t \, \mathrm{d}x = 0$$

which implies (3.9) as soon as we have (3.10).

Let us now prove (3.10). We have solved in Theorem 2.5 the linear problem with source term: there is a unique C^1 solution given by

$$\varphi(t,x) = \varphi_0(Z_{t,0}(x)) + \int_0^t \psi(s, Z_{t,s}(x)) \, \mathrm{d}s.$$

What remains to be proved in order to solve (3.10) is that, by playing with the different initial data φ_0, we can build a compactly supported solution. Let us define

$$\forall x \in \mathbb{R}, \quad \varphi_0(x) := -\int_0^T \psi(s, Z_{0,s}(x)) \, \mathrm{d}s. \tag{3.11}$$

This is a C^1 compactly supported initial data, and the associated solution to (3.9) is

$$\begin{aligned} \varphi(t,x) =& \varphi_0(Z_{t,0}(x)) + \int_0^t \psi(s, Z_{t,s}(x)) \, \mathrm{d}s \\ =& -\int_0^T \psi(s, Z_{0,s} \circ Z_{t,0}(x)) \, ds + \int_0^t \psi(s, Z_{t,s}(x)) \, \mathrm{d}s \\ =& \int_T^t \psi(s, Z_{t,s}(x)) \, \mathrm{d}s. \end{aligned}$$

We know that ψ has compact support, say included in $[0,T] \times [-R,R]$. Then for $t \geq T$, the function φ vanishes since the integrand above vanishes. Second, for $t \in [0,T]$, one has $|t-s| \in [0,T]$ since $0 \leq t \leq s \leq T$, and therefore $\varphi(t,x)$ vanishes as soon as $|Z_{t,s}(x)| \geq R$ for all s, which is true as soon as $|x| \geq R + LT$. Finally we deduce that φ has compact support in $[0,T] \times [-R-LT, R+LT]$. This concludes the proof. □

REMARK 3.13. *One could prove the existence and uniqueness of weak solutions with a source term as well with the same methods.*

4. Nonlinear scalar monodimensional transport equations

Let us consider the simpler nonlinear first-order PDE, where the nonlinearity only depends on the solution itself and not on time or space:

$$\begin{cases} \partial_t u + \partial_x \left[f(u)\right] = 0, \quad t \geq 0, \ x \in \mathbb{R} \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{4.1}$$

where f is called the *flux* of the transport equation. When both u and f are C^1, the equation writes also

$$\partial_t u + f'(u)\, \partial_x u = 0 \tag{4.2}$$

$c(t,x) = f'(u(t,x))$ corresponds to the speed of propagation as seen before. However this speed now depends on the value of the solution itself $u(t,x)$ at the time and point considered. Unlike the previous linear transport equations that were completely understood through the ODE theory, we shall encounter a phenomenon genuinely specific to transport PDEs.

4.1. Classical solutions and characteristics. As before we first define the notion of solutions.

DEFINITION 4.1. *Consider $f \in C^2(\mathbb{R})$ with $f' \in L^\infty(\mathbb{R})$, and $u_0 \in C^1(\mathbb{R})$ with $u_0, u_0' \in L^\infty(\mathbb{R})$. We say that $u = u(t,x)$, $t \in [0,T]$, $x \in \mathbb{R}$, is a* classical solution *to* (4.1) *on $[0,T]$, with initial data u_0, if*

- *u is $C^1([0,T] \times \mathbb{R})$ with $u, \partial_x u \in L^\infty(\mathbb{R})$;*
- *u satisfies* (4.1) *as an equality between continuous functions.*

We now consider a solution a priori given, and apply the previous notion of characteristics seen for scalar transport equations with variable coefficients, to the case of $c(t,x) = f'(u(t,x))$:

DEFINITION 4.2. *We call* characteristics of the transport equation (4.1)*, the trajectories $(Z_{s,t})$, $0 \le s,t < +\infty$, of the differential system:*

$$\begin{cases} \partial Z_{s,t}(x) = c(t, Z_{s,t}(x)) = f'\left(u(t, Z_{s,t}(x)\right), \quad s,t \ge 0,\ x \in \mathbb{R}^d, \\ Z_{s,s}(x) = x, \quad x \in \mathbb{R}^d. \end{cases}$$

PROPOSITION 4.3. *Under the previous assumptions on f and the solution u in the definition of classical solutions, the characteristics maps above exist, are unique and C^1 on $[0,T]$, moreover for any $s,t \in [0,T]$,*

$$Z_{s,t} : \mathbb{R}^d \to \mathbb{R}^d, \quad x \mapsto Z_{s,t}(x)$$

is a C^1-diffeomorphism.

PROOF. The proof is similar as for variable coefficients (application of Picard-Lindelöf). □

Let us now reason heuristically on our a priori solution (or to be precise, we shall reason by necessary conditions). The key remark to come is at the origin of the existence and uniqueness theorem, but also of the limits of this construction of classical solutions.

As before the solution u is *constant along the characteristic trajectories* (except that now these trajectories depend on the solution itself)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[u(t, Z_{0,t}(x))\right] &= (\partial_t u)(t, Z_{0,t}(x)) + \partial_t Z_{0,t}(y)\ (\partial_x u)(t, Z_{0,t}(x)) \\ &= (\partial_t u)(t, Z_{0,t}(x)) + f'(u(t, Z_{0,t}(x)))\ (\partial_x u)(t, Z_{0,t}(x)) = 0 \end{aligned}$$

since u satisfies the equation at the point $(t, Z_{0,t}(x))$. We deduce that

$$\forall\, t \in [0,T],\ x \in \mathbb{R}, \quad u(t, Z_{0,t}(x)) = u(0, Z_{0,0}(x)) = u(0,x) = u_0(x).$$

Coming back to the differential equation defining the characteristic trajectories, we have for $x \in \mathbb{R}$ fixed:

$$\forall\, t \ge 0, \quad \partial_t Z_{0,t}(x) = f'(u(t, Z_{0,t}(x)) = f'(u_0(x))$$

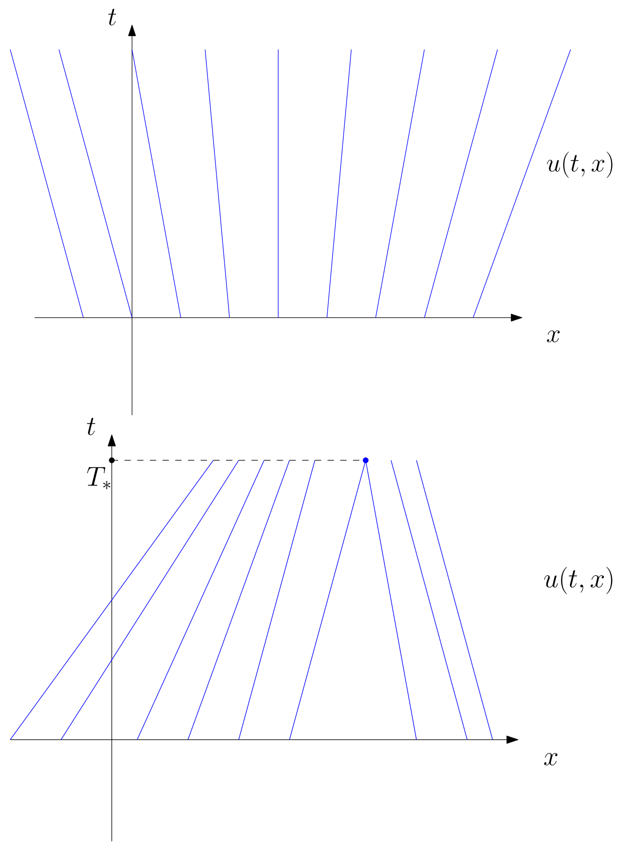

FIGURE 4.1. Picture of characteristics in the cases $f' \circ u_0$ increasing and then decreasing.

which implies that the characteristics are in fact lines: $Z_{0,t}(x) = x + tf'(u_0(x))$ (their time derivative is constant). However, unlike the case of a constant coefficient in section 2, the slope of the line varies according to x, the starting point of the characteristic, along with the function $f' \circ u_0$.

From the previous study of the linear transport equation with variable coefficient, we expect the solution to be given by $u(t,x) = u_0(Z_{t,0}(x))$. However, it is clear from the picture that in the second case there will be a problem at time T_*: two potentially different values of the initial data should become equal if this characteristic formula would remain true. We shall now characteristize the first time when characteristic lines cross, and construct classical solutions until this time.

4.2. Classical solution for short times. The main theorem is:

THEOREM 4.4. *Consider $f \in C^2(\mathbb{R})$ with $f' \in L^\infty(\mathbb{R})$, and $u_0 \in C^1(\mathbb{R})$ with $u_0, u_0' \in L^\infty(\mathbb{R})$, and define $T_* \in \mathbb{R}_+ * \cup \{+\infty\}$ by (1) $T_* := +\infty$ if $f' \circ u_0$ is non-decreasing, or else (2)*

$$T_* := -\left[\min_{x\in\mathbb{R}} \left(f' \circ u_0\right)'\right]^{-1}. \tag{4.3}$$

Then there exists a unique classical solution u to (4.1) *with initial data u_0 sur $[0, T_*)$, i.e. on any interval $[0, T]$ for $0 < T < T_*$.*

REMARKS 4.5. (1) *The time T_* is precisely the first time when characteristic lines cross. Indeed let us consider some time t when such crossing occurs: one has $x_0 < x_1$ so that*

$$t + (f' \circ u_0)(x_0) = t + (f' \circ u_0)(x_1)$$

and thus

$$t = -\left[\frac{(f' \circ u_0)(x_0) - (f' \circ u_0)(x_1)}{x_0 - x_1}\right]^{-1}.$$

By the mean value theorem we deduce that

$$t = -\left[(f' \circ u_0)'(x_2)\right]^{-1}$$

for some $x_2 \in [x_0, x_1]$, and thus $t \geq T_*$. On the other hand, if $t > T_*$, then $x \mapsto Z_{0,t}(x)$ is not monotonic (see below), and thus not injective, and there exist $x_0 < x_1$ so that $Z_{0,t}(x_0) = Z_{0,t}(x_1)$, which corresponds to a crossing.

(2) *When applied to the Burgers equation, we have $f'(x) = x$ and the critical time is $T_* = +\infty$ if u_0 is non-decreasing, and else*

$$T_* = -\left[\min_{x\in\mathbb{R}} u_0'\right]^{-1}.$$

PROOF. We separate existence and uniqueness as before.

Uniqueness: Consider a classical solution u on $[0, T]$ with $0 < T < T_*$ and initial data u_0. From the previous proposition we can define (in a unique manner) the characteristic maps $x \mapsto Z_{0,t}(x)$ for any $t \in [0, T]$ and $x \in \mathbb{R}$, which are C^1-diffeomorphisms. We then compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[u(t, Z_{0,t}(x))\big] = 0 \quad \Longrightarrow \quad u\left(t, Z_{0,t}(x)\right) = u_0(x)$$

and thus $u(t, x) = u_0\left(Z_{t,0}(x)\right)$ (explicit characteristic formula). Coming back to the characteristics ODEs we get

$$\partial_t Z_{0,t}(x) = f'(u(t, Z_{0,t}(x))) = f'\left(u\left(t, Z_{0,t}(x)\right)\right) = f'\left(u_0(x)\right)$$

and therefore

$$Z_{0,t}(x) = x + t f'\left(u_0(x)\right). \tag{4.4}$$

Let us check now (even if it predicted by Picard-Lindelöf) that these maps are C^1-diffeomorphisms, as this will explain the definition of T_*. This reduces to verify the monotonicity:

$$\forall\, t \in [0, T], \quad Z_{0,t}'(x) = 1 + t\big(f' \circ u_0\big)'(x) \geq 1 + t\Big[\min_{y\in\mathbb{R}} \big(f' \circ u_0\big)'(y)\Big] \geq 1 - \frac{t}{T_*} > 0$$

which is uniformly positive on $\mathbb{R}$. Note that the case $T_* = +\infty$ is included as a particular case of the previous formula. Note also the critical role played by T_* when it is finite. The explicit characteristic formula, together with (4.4) and the fact that $Z_{0,t}$ is a C^1-diffeomorphism for $t \in [0, T]$, characterizes entirely the solution, and proves uniqueness.

Existence: From the previous discussion, it is natural to define a candidate solution w on $[0,T]$ as

$$w(t,x) = u_0\left(Z_{t,0}(x)\right)$$

together with the formulas

$$Z_{0,t}(x) = x + tf'(u_0(x)).$$

This definition makes sense: we can invert $Z_{0,t}$ from the calculation we have made above to check that it is a diffeomorphism. The function w is $C^1([0,T]\times\mathbb{R})$ and by chain-rule

$$\partial_x w(t,x) = \frac{u_0\left(Z_{t,0}(x)\right)}{Z'_{0,t}\left(Z_{t,0}(x)\right)}$$

and thus

$$|\partial_x w(t,x)| \le \frac{\|u_0\|}{1-(t/T_*)}$$

which proves that $\partial_x w \in L^\infty(\mathbb{R})$. We also have the initial condition $w(0,x) = u_0(x)$ since $Z_{0,0}(x) = x$.

By inverting the characteristic map we get $w(t,Z_{0,t}(x)) = u_0(x)$ and by differentiating in time:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[w\left(t,Z_{0,t}(x)\right)\right] = \left(\partial_t w\right)\left(t,Z_{0,t}(x)\right) + \left(\partial_t Z_{0,t}(x)\right)\,\left(\partial_x w\right)\left(t,Z_{0,t}\right).$$

Since $\partial_t Z_{0,t}(x) = f'(u_0(x))$, we get

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[w\left(t,Z_{0,t})\right)\right] = \left(\partial_t w\right)\left(t,Z_{0,t})(x)\right) + f'\left(u_0(x)\right)\,\left(\partial_x w\right)\left(t,Z_{0,t}(x)\right).$$

By using again the characteristic formula, we have $f'(u_0(x)) = f'(w(t,Z_{0,t}(x)))$, and therefore

$$\forall\, t\in[0,T],\ x\in\mathbb{R},\quad \left(\partial_t w\right)\left(t,Z_{0,t}(x)\right) + f'\left(w\left(t,Z_{0,t}(x)\right)\right)\,\left(\partial_x w\right)\left(t,Z_{0,t}(x)\right) = 0.$$

Hence the equation is satisfied for any $t\in[0,T]$, and $y = Z_{0,t}(x)$, $x\in\mathbb{R}$. Since $Z_{0,t}$ is bijective, this concludes the proof. □

REMARK 4.6. *Let us observe a fundamental property of transport equation (and in fact more generally hyperbolic equations): finite speed of propagation of information: if u_0 has compact support included in $[-B,B]$, then the solution $u(t,\cdot)$ at time t has also compact support included*

$$\left[-B + t\min_{x\in\mathbb{R}} f'(u_0(x)), B + t\max_{x\in\mathbb{R}} f'(u_0(x))\right].$$

This is the notion of cone of dependency*. More generally, the values of the solution $u(t,\cdot)$ at time t on some compact interval $[-A,A]$ only depends on the values of the initial data on the compact interval*

$$\left[-A + t\min_{x\in\mathbb{R}} f'(u_0(x)), A + t\max_{x\in\mathbb{R}} f'(u_0(x))\right].$$

This is the notion of cone de influence*. The proofs are let as an exercise.*

4.3. Non-existence of global classical solutions. Let us now study what happens at the critical time T_*, and prove that derivatives are diverging, so that it is not possible to extend further C^1 classical solutions.

THEOREM 4.7. *Consider $f \in C^2(\mathbb{R})$ with $f' \in L^\infty(\mathbb{R})$, $u_0 \in C^1(\mathbb{R})$ with $u_0, u_0' \in L^\infty(\mathbb{R})$, and define $T_* \in \mathbb{R}_+ * \cup \{+\infty\}$ as before. If $T_* < +\infty$, then there is no classical solution to* (4.1) *with initial data u_0 on a time interval $[0,T]$ with $T > T_*$.*

PROOF OF THEOREM 4.7. Let us give two proofs, each one interesting in its own right for the mechanism of formation of shocks. We assume by contradiction the existence of a classical solution u on $[0,T]$ with $T > T_*$ and initial data u_0. From the uniqueness of classical solutions on $[0,T_*)$, it coincides with the solution constructed above by the characteristic method on $[0,T_*)$.

First poof: For $0 < t < T_*$ we have

$$u(t,x) = u_0(Z_{t,0}(x))$$

and compute

$$\partial_x u(t,x) = u_0'(Z_{t,0}(x))\, Z_{t,0}'(x) = \frac{u_0'(Z_{t,0}(x))}{Z_{0,t}'(Z_{t,0}(x))} = \frac{u_0'(Z_{t,0}(x))}{1 + t\,(f'(u_0))'(Z_{t,0}(x))}.$$

From the definition of T_*, there is $x(t) \in \mathbb{R}$, $t \to T_*$ so that

$$1 + t(f'(u_0))'(Z_{t,0}(x(t))) \to 0^+, \quad t \to T_*$$

which writes

$$(f'(u_0))'(Z_{t,0}(x(t))) = f''(u_0(Z_{t,0}(x(t))))\, u_0'(Z_{t,0}(x(t))) \to -\frac{1}{T_*}.$$

The fact that $u_0 \in L^\infty$ and $f'' \in C^0$ imply

$$\forall\, t < T_*, \quad \left|f''(u_0(Z_{t,0}(x(t))))\right| \le M < +\infty$$

and thus

$$\left|u_0'(Z_{t,0}(x(t)))\right| \ge \frac{1}{2\,M\,T_*} > 0.$$

We deduce that

$$\left|\partial_x u(t,x(t))\right| \ge \frac{1}{2\,M\,T_*}\, \frac{1}{\left|1 + t\,(f'(u_0))'(Z_{t,0}(x(t)))\right|} \to +\infty \qquad \text{as} \qquad t \to T_*.$$

This contradicts $\partial_x u \in L^\infty$ and concludes the proof.

Second proof (and precise blow-up behavior): We differentiate in x the equation to write an evolution equation on $\partial_x u$ on $[0,T_*)$:

$$\partial_t(\partial_x u) + f'(u)\,\partial_x(\partial_x u) + f''(u)\,(\partial_x u)^2 = 0.$$

The first two terms have the same structure of the original transport equation, and the last term can be seen as a source term, leading (by Duhamel principle) to the following formula:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\partial_x u(t, Z_{0,t}(x))\right] &= \left[\partial_t(\partial_x u) + f'(u)\,\partial_x(\partial_x u)\right]_{(t, Z_{0,t}(x))} \\ &= -f''(u(t, Z_{0,t}(x)))\,(\partial_x u(t, Z_{0,t}(x)))^2.\end{aligned}$$

Since u is constant along characteristic trajectories, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\partial_x u(t, Z_{0,t}(x))\right] = -f''(u_0(x))\,(\partial_x u(t, Z_{0,t}(x)))^2.$$

If we define

$$w(t, x) := f''(u_0(x))\,\partial_x u(t, Z_{0,t}(x))$$

we thus get the following closed on w:

$$\partial_t w(t, x) = -w(t, x)^2.$$

This is continuum of ODEs in time, indexed by the space variable x. Solving explicitely this equation yields

$$w(t, x) = \frac{1}{t - T(x)} \quad \text{with} \quad T(x) := -\frac{1}{w(0, x)} = -\frac{1}{(f' \circ u_0)'(x)}.$$

(One excludes the case $w(0, x) = 0$ which yields the zero solution.)

We deduce that if $w(0, x) \geq 0$, then the solution is global ($T(x) \leq 0$), with moreover the decay estimate

$$\forall\, t \geq 1, \quad |w(t, x)| \leq \frac{1}{t}$$

meaning on the original solution

$$\forall\, t \geq 1, \quad |\partial_x u(t, Z_{0,t}(x))| \leq \frac{1}{|f''(u_0(x))|\, t}.$$

On the other hand if $w(0, x) < 0$, then $T(x) > 0$ and $w(t, x) \to +\infty$ as $t \to T(x)$. Finally observe that from the definitions of $T(x)$ and T_* we have

$$T_* = \inf_{x \in \mathbb{R},\ T(x) > 0} T(x).$$

This concludes the proof. □

Remark 4.8. *The second proof shows that the mechanism for "blow-up" (i.e. formation of singularity and blow-up of the first derivative) is at the same time more complex than for ODEs as it involves the nonlinear transport evolution of two different values at the same point, but has an underlying "ODE blow-up mechanism" for the amplitude of this first derivative, when properly reframed along the characteristics.*

Let us now prove that when f is nonlinear there is *generically* no global classical solutions.

Theorem 4.9. *Consider $f \in C^2(\mathbb{R})$ with f' bounded and not constant, then there exists $u_0 \in C^1(\mathbb{R})$ with $u_0, u_0' \in L^\infty(\mathbb{R})$ so that the unique classical solution* (4.1) *with initial data u_0 breaks down in finite time as described above ($T_* < +\infty$).*

Proof of Theorem 4.9. It is enough to construct $u_0 \in C^1$ with compact support so that $f'(u_0)$ is not non-decreasing and then apply the previous theorem for this choice of initial data (for which $T_* < +\infty$). The construction of such a u_0 is let as an exercise. □

4.4. Weak L^∞ and entropic solutions. The previous discussion motivates the introduction of a notion of weak solutions, since classical solutions are not enough.

DEFINITION 4.10. *Consider $f \in C^1(\mathbb{R})$ with $f \in L^\infty(\mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R})$, we say that $u = u(t,x)$, $t \geq 0$, $x \in \mathbb{R}$ is a weak solution to* (4.1) *on* $[0,T]$ *if $u \in L^\infty([0,T] \times \mathbb{R})$ and*

(4.5)

$$\forall\, \varphi \in C^1_c([0,T) \times \mathbb{R}), \quad \int_0^T \int_{\mathbb{R}} \big(u\, \partial_t \varphi + f(u)\, \partial_x \varphi\big)\, \mathrm{d}t\, \mathrm{d}x + \int_{\mathbb{R}} \varphi(0,x) u_0(x)\, \mathrm{d}x = 0.$$

REMARKS 4.11.
1. *This is the same definition as in the linear case, only paying attention to the nonlinear term $f(u)$.*
2. *One can prove that the definition is equivalent to the same statement replacing the space of test functions by $C^\infty_c(\mathbb{R}^*_+ \times \mathbb{R})$. This shows that this weak formulation implies the equation $\partial_t u + \partial_x f(u) = 0$ sur $[0,T] \times \mathbb{R}$ in distributional sense.*

We leave as an exercise the tedious but necessary step of checking the consistency of this definition with of classical solutions.

PROPOSITION 4.12. *Consider $f \in C^2(\mathbb{R})$ with $f' \in L^\infty(\mathbb{R})$ and $u_0 \in C^1(\mathbb{R})$ with $u_0, u_0' \in L^\infty(\mathbb{R})$. Then (1) the classical solution $u(t,x)$ on $[0,T]$ (if it exists) is also a weak solution, (2) any weak solution $u(t,x)$ on $[0,T]$ that is also $C^1([0,T] \times \mathbb{R})$ is a classical solution.*

Observe that the same result can be done for the more general equation $\partial_t u + \partial_x q = 0$ on the pair (u,q) in an open set $\mathcal{U} \subset [0,T] \times \mathbb{R}$. The weak formulation is defined simply as $u, q \in L^\infty(\mathcal{U})$ with

$$\forall\, \varphi \in C^1_c(\mathcal{U}), \quad \int_{\mathcal{U}} (u \partial_t \varphi + q \partial_x \varphi)\, \mathrm{d}t\, \mathrm{d}x = 0,$$

and we have (1) if a classical solution (u,q) exists on $\mathcal{U}$ then it is also a weak solution, (2) any weak solution (u,q) on $\mathcal{U}$ that is also $C^1(\mathcal{U})$ is a classical solution.

Let us see that the definition of weak solution is already sufficient to derive an important requirement on any isolated curve of discontinuity, called the *Rankine-Hugoniot* condition.

PROPOSITION 4.13. *Let us consider a pair (u,q) of functions, piecewise continuous in the domain $\mathcal{U}$, whose line of discontinuity lies along a regular curve Γ, which separates $\mathcal{U}$ into two connected components $\mathcal{U}_\pm$. We assume that (u,q) is of class C^1 in $\mathcal{U}_-$ and in $\mathcal{U}_+$, where it satisfies $\partial_t u + \partial_x q = 0$. Finally, we denote by $u_+(x,t)$ the limit of $u(y,s)$ when (y,s) tends to $(x,t) \in \Gamma$ and stays in $\mathcal{U}_+$. In the same way we define $q_+(x,t)$ and $u_-(x,t)$ and $q_-(x,t)$ along Γ, and we write $[h](x,t) = h_+(x,t) - h_-(x,t)$, the jump across Γ of any piecewise continuous function h.*

Under the above hypothese, the pair (u,q) satisfy the equation in the distributional sense in $\mathcal{U}$ if and only if

1. *On the one hand, u and q satisfy the equation pointwise in $\mathcal{U}_+$ and $\mathcal{U}_-$.*

(2) On the other hand, the jump condition

$$\forall\,(t,x) \in \Gamma, \quad [u](t,x)n_t(t,x) + [q](t,x)n_x(t,x) = 0$$

is satisfied along Γ*, where* $\mathbf{n} = (n_t, n_x)$ *is a unit normal vector to* Γ *in* (x,t)*.*

Proof. Let (u,q) be a solution of the weak formulation on $\mathcal{U}$ and with the regularity assumptions. First of all choosing test functions whose support is in $\mathcal{U}_-$, we see that (u,q) is a weak solution in $\mathcal{U}_-$. In the same way we have the result for $\mathcal{U}_+$. By the weak-strong uniqueness result above this implies that (u,q) satisfy the equation in the classical sense in $\mathcal{U}_\pm$.

We then calculate with the Green theorem:

$$\begin{aligned} 0 &= \int_{\mathcal{U}} (u\partial_t\varphi + q\partial_x\varphi)\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_{\mathcal{U}_+} (u\partial_t\varphi + q\partial_x\varphi)\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathcal{U}_-} (u\partial_t\varphi + q\partial_x\varphi)\,\mathrm{d}t\,\mathrm{d}x \\ &= -\int_{\mathcal{U}_+} \varphi\,(\partial_t u + \partial_x q)\,\mathrm{d}t\,\mathrm{d}x - \int_{\mathcal{U}_-} \varphi\,(\partial_t u + \partial_x q)\,\mathrm{d}t\,\mathrm{d}x \\ &\quad \int_{\partial\mathcal{U}_+} \varphi\left(u_+ n_t^+ + q_+ n_x^+\right)\mathrm{d}s + \int_{\partial\mathcal{U}_-} \varphi\left(u_- n_t^- + q_- n_x^-\right)\mathrm{d}s. \end{aligned}$$

We use that the two first term are zeros from the previous step, and that $\mathbf{n}^+ = -\mathbf{n}^-$, to get

$$\int_\Gamma ([u]n_t + [q]n_x)\,\mathrm{d}s = 0$$

which implies $[u]n_t + [q]n_x = 0$ on Γ as it is true for any $\varphi \in C_c^1(\mathcal{U})$.

The other implication is proved similarly. □

When $q = f(u)$, the jump condition is called the *RankineHugoniot condition.* If the curve of discontinuity writes as

$$\Gamma = \{(X(t),t) : t \in I\}$$

then it takes the form

$$[f(u)] = X'(t)[u].$$

By the mean value theorem this implies that

$$X'(t) = c(\bar{u}(t)) = f'(\bar{u}(t)) \qquad \text{where} \qquad \bar{u}(t) \in [u_-(t,X(t)), u_+(t,X(t)]$$

so the slope of the curve lies between the left and right speeds of propagation. When the amplitude of discontinuity approaches zero, this slope approaches the exact propagation speed of the characteristic trajectory at the point.

The simplest discontinuous solutions are of the form $u = u_-$, $x < \sigma t$, $u = u_+$, $x > \sigma t$, where $\sigma = (f(u+) - f(u_-))/(u_+ - u_-)$. Indeed, u satisfies the equation trivially outside of the straight line $x = \sigma t$. For Burgers' equation $f(u) = u^2/2$, the speed of propagation of the discontinuities is $X'(t) = (u_+ + u_-)/2$.

This result is actually enough to entirely characterize the structure of weak solutions that are piecewise constant. And this allows us to build easily such weak solutions and show that uniqueness does not hold. Consider the Burgers equation with zero initial condition $u_0 = 0$. We have then of course the trivial global weak solution $u = 0$. Consider then the following function for any parameter $p > 0$

$$u(t,x) = \begin{cases} 0, & x < -pt \\ -2p, & -pt < x < 0 \\ 2p, & 0 < x < pt \\ 0, & pt < x. \end{cases}$$

On can check that the equation is satisfied in the classical sense in each four region, and that the Rankine-Hugoniot are satisfied at the interfaces. This proves that u is a weak solution. We have therefore built an infinite number of solutions. The Riemann problem, that is the Cauchy problem starting from a two-values discontinuous initial condition will be studied in more details in the example classes.

Hence we see that the definition of L^∞ weak solutions lead to *infinitely* many solutions. The correct definition adds a further condition, inspired from the second principle of thermodynamics, asking that informations flows forward in time, or here more precisely that characteristics enter the shocks as time moves forward, and never comes out out of the shocks.

Definition 4.14. *Consider $f \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ et $u_0 \in L^\infty(\mathbb{R})$. We say that $u = u(t,x)$, $t \geq 0$, $x \in \mathbb{R}$ is an entropic solution to* (4.1) *on* $[0,T)$*,* $T \in \mathbb{R}_+ \cup \{+\infty\}$*, if $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ and*

$$\forall\, \varphi \in C^1_c([0,T) \times \mathbb{R}; \mathbb{R}_+),\ \eta \in C^1(\mathbb{R};\mathbb{R}) \text{ convex},\ \ \phi \text{ s.t. } \phi' = f'\eta',$$

$$\int_0^T \int_{\mathbb{R}} \big(\eta(u)\partial_t\varphi + \phi(u)\partial_x\varphi\big)\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} \varphi(0,x)\eta(u_0(x))\,\mathrm{d}x \geq 0. \tag{4.6}$$

Remark 4.15. *Pay attention to the fact that the test functions are assumed to be non-negative here.*

We first check the consistency as before (the proof is left as an exercise).

Proposition 4.16. *Any entropic solution is a weak solution. Any classical solution is an entropic solution. Any C^1 entropic solution is a classical solution.*

Another important consistency checking is that the entropic conditions follow from the *vanishing viscosity approximation*:

Exercise 48. *Consider the equation*

$$\partial_t u + \partial_x f(u) = \varepsilon \partial^2_{xx} u$$

with initial data $u_0 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. One can show that this PDE admits a unique global solution $u_\varepsilon \in C^1(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R})$. Prove that

(1) if $u_\varepsilon \to u$ almost everywhere then u is a weak solution to the transport equation $\partial_t u + \partial_x f(u) = 0$ with initial condition u_0;

(2) it is moreover an entropic solution.

Hence the inequalities with the entropic-flux pairs in the definition of entropic solution can be understood as the time-arrow information that should be retained from the microscopic dissipative mechanisms that are neglected (other dissipative approximations *would give the same results). They have the effect to prevent characteristics getting out from shocks (discontinuity curves).*

We end up with an important and beautiful theorem. We shall prove it in dimension 1 but it remains true in higher dimensions. However it does not apply to systems of transport equations, which are much less well understood.

THEOREM 4.17 (Kružkov, 1970). *Consider $f \in C^1(\mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R})$. Then there exists a unique entropic solution to* (4.1) *with initial data u_0. This solution u moreover satisfies $u \in C(\mathbb{R}_+, L^1_{loc}(\mathbb{R}))$ and $\|u\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} = \|u_0\|_{L^\infty(\mathbb{R})}$.*

REMARK 4.18. *Observe that this unique entropic solution can no more be determined simply by computing the characteristic curves. We have to leave the pure Lagrangian approach and go back to a (more modern) energy method.*

PROOF. We shall admit here the existence part, which can be obtained by approximation methods. We shall prove the uniqueness part, in the form of a powerful and enlighting contraction inequality.

Consider for any $k \in \mathbb{R}$ the function $\eta(u) = |u-k|$ which is convex, with associated flux $\phi(u) = (f(u) - f(k))\mathrm{sgn}(u-k)$. Applying the definition of entropic solutions with this pair entropy-flux gives the following inequality

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(\partial_t\varphi|u-k| + \partial_x\varphi(f(u)-f(k))\mathrm{sgn}(u-k)\right) \mathrm{d}t\,\mathrm{d}x + \int |u_0-k|\varphi(0,x)\,\mathrm{d}x \geq 0.$$

We consider now two entropic solutions u, v and a test function $\Phi(t,x,s,y) \geq 0$ smooth and compactly supported in $([0,T)\times\mathbb{R})^2$, and apply this previous inequality first on u with $k = v(s,y)$ and then on v with $k = u(t,x)$ (this is the *doubling of variables method*) and then integrate in the remaining variables and sum these two inequalities. We obtain

$$\begin{aligned}
0 \leq &\int_0^T\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} |u(t,x)-v(s,y)|(\partial_t\Phi+\partial_s\Phi)\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y\\
&+\int_0^T\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{sgn}(u(t,x)-v(s,y))[f(u(t,x))-f(v(s,y))](\partial_x\Phi+\partial_y\Phi)\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y\\
&+\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} |u_0(x)-v(s,y)|\Phi(0,x,s,y)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y\\
&+\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} |u(t,x)-v_0(y)|\Phi(t,x,0,y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

We choose the test function as

$$\Phi(t,x,s,y) := \varphi(t,x)\chi_\varepsilon(t-s,x-y), \quad \chi_\varepsilon(\tau,z) := \varepsilon^{-2}\chi\left(\frac{\tau}{\varepsilon},\frac{z}{\varepsilon}\right), \quad \chi(\tau,z) := \eta(\tau)\theta(z)$$

with $\eta \geq 0$ smooth with mass one and support in $[-2,-1]$ and $\theta \geq 0$ smooth with mass one and compact support. Observe that

$$\partial_t\Phi + \partial_s\Phi = (\partial_t\varphi)\chi_\varepsilon, \quad \partial_x\Phi + \partial_y\Phi = (\partial_x\varphi)\chi_\varepsilon.$$

We then pass to the limit $\varepsilon \to 0$ in the previous integrals, and claim that

$$\int_0^T \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} |u(t,x) - v(s,y)| (\partial_t \varphi) \chi_\varepsilon \,\mathrm{d}t \,\mathrm{d}s \,\mathrm{d}x \,\mathrm{d}y \xrightarrow[\varepsilon \to 0]{} \int_0^T \int_{\mathbb{R}} |u(t,x) - v(t,x)| \partial_t \varphi \,\mathrm{d}t \,\mathrm{d}x$$

$$\int_0^T \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{sgn}(u(t,x) - v(s,y))[f(u(t,x)) - f(v(s,y))](\partial_x \varphi)\chi_\varepsilon \,\mathrm{d}t \,\mathrm{d}s \,\mathrm{d}x \,\mathrm{d}y \xrightarrow[\varepsilon \to 0]{}$$

$$\int_0^T \int_{\mathbb{R}} \mathrm{sgn}(u(t,x) - v(t,x))[f(u(t,x)) - f(v(t,x))] \partial_x \varphi \,\mathrm{d}t \,\mathrm{d}x$$

$$\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} |u_0(x) - v(s,y)| \varphi(0,x) \chi_\varepsilon(-s, x-y) \,\mathrm{d}s \,\mathrm{d}x \,\mathrm{d}y \xrightarrow[\varepsilon \to 0]{} \int_{\mathbb{R}} |u_0(x) - v_0(x)| \varphi(0,x) \,\mathrm{d}x$$

$$\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} |u(t,x) - v_0(y)| \varphi(t,x) \chi_\varepsilon(t, x-y) \,\mathrm{d}s \,\mathrm{d}x \,\mathrm{d}y = 0.$$

The last term is zero for all ε because of the support condition on η. Let us prove for instance the first limit. We write (using the mass condition on χ_ε):

$$\int_0^T \int_{\mathbb{R}} |u(t,x) - v(t,x)| \partial_t \varphi \,\mathrm{d}t \,\mathrm{d}x = \int_0^T \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} |u(t,x) - v(t,x)| \partial_t \varphi \chi_\varepsilon \,\mathrm{d}t \,\mathrm{d}s \,\mathrm{d}x \,\mathrm{d}y$$

and therefore the claimed convergence reduces (by triangular inequality) to prove that

$$I_\varepsilon := \int_0^T \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} |v(s,y) - v(t,x)| \partial_t \varphi \chi_\varepsilon \,\mathrm{d}t \,\mathrm{d}s \,\mathrm{d}x \,\mathrm{d}y \xrightarrow[\varepsilon \to 0]{} 0.$$

By change of variable this latter integral writes

$$I_\varepsilon = \int_0^T \int_{\dots} \int_{\mathbb{R}} \int_{\mathbb{R}} |v(t + \varepsilon\tau, x + \varepsilon z) - v(t,x)| \partial_t \varphi(t,x) \chi(\tau, z) \,\mathrm{d}t \,\mathrm{d}\tau \,\mathrm{d}x \,\mathrm{d}z.$$

From the compact support and the continuity of v, we deduce by the dominated convergence theorem that $I_\varepsilon \to 0$.

Exercise 49. *Prove that the convergence of the other terms can be treated similarly.*

We therefore deduce that for all $0 \le \varphi \in C_c^1$

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} |u(t,x) - v(t,x)| \partial_t \varphi \,\mathrm{d}t \,\mathrm{d}x \\ &+ \int_0^T \int_{\mathbb{R}} \mathrm{sgn}(u(t,x) - v(t,x))[f(u(t,x)) - f(v(t,x))] \partial_x \varphi \,\mathrm{d}t \,\mathrm{d}x \\ &+ \int_{\mathbb{R}} |u_0(x) - v_0(x)| \varphi(0,x) \,\mathrm{d}x \ge 0. \end{aligned}$$

We shall now perform an energy estimate along the cone of dependency, which is a fundamental idea for hyperbolic equations. Consider two solutions $u, v \in L^\infty$, $[a,b] \subset \mathbb{R}$ (say w.l.o.g $-a = b > 0$) and $s \in (0,T)$ and $M = \sup_{[-C,C]} |f'|$ with $C = \max(\|u\|_\infty, \|v\|_\infty)$ the maximal speed of wave propagation. Consider the trapezium

$$B := \{ t \in [0,s],\ a - M(t-s) < x < a + M(t-s) \}.$$

Consider $\theta = \theta(r) \ge 0$ a smooth function on $\mathbb{R}_+$ so that $\theta = 1$ on $[0,b]$ and $\theta = 0$ oustide $[0, b+\varepsilon]$, and $0 \le \chi \in C_c^1([0,T))$ so that so that $\chi(0) = 1$ and $\chi(t) = 0$ so

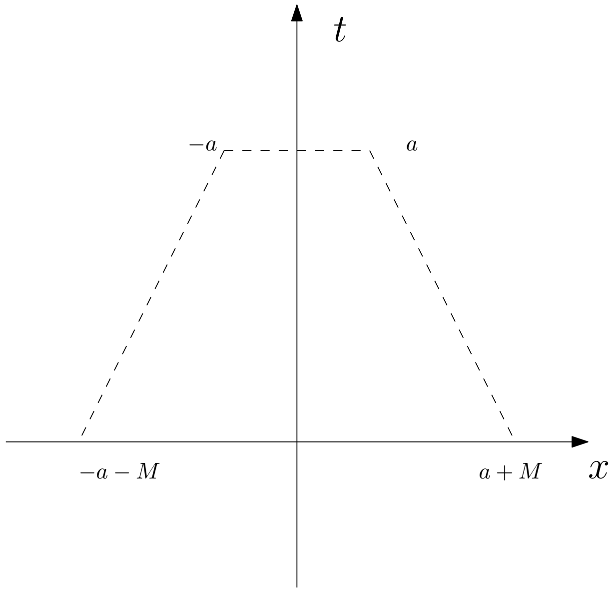

FIGURE 4.2. Trapezium B for the energy estimate.

$t > s'$ with $s < s' < s + b/M$, $s, s' \in [0, T)$. We apply the previous inequality with the test function $\varphi(t,x) = \chi(t)\theta(|x| + Mt)$, with the observation (with the notation $F(u,v) = \mathrm{sgn}(u(t,x) - v(t,x))[f(u(t,x)) - f(v(t,x)))$

$$\begin{aligned}
&|u(t,x) - v(t,x)|\partial_t\varphi + F(u,v)\partial_x\varphi = |u(t,x) - v(t,x)|\chi'(t)\theta(|x| + Mt)\\
&\qquad + \Big(F(u,v)\mathrm{sgn}(x) + M|u(t,x) - v(t,x)|\Big)\chi(t)\theta'(|x| + Mt)\\
&\leq |u(t,x) - v(t,x)|\chi'(t)\theta(|x| + Mt)
\end{aligned}$$

since $|F(u,v)| \leq M|u(t,x) - v(t,x)|$ and $\theta' \leq 0$. By integration, we get

$$\int_0^T \int_{\mathbb{R}} |u(t,x) - v(t,x)|\chi'(t)\theta(|x| + Mt)\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} |u_0(x) - v_0(x)|\theta(|x|)\,\mathrm{d}x \geq 0$$

(where we have used $\chi(0) = 1$). By taking θ converging to the characteristic function of the domain $[a, b]$, we deduce by the dominated convergence theorem

$$\int_0^T \int_{a-M(t-s)}^{b+M(t-s)} |u(t,x) - v(t,x)|\chi'(t)\,\mathrm{d}t\,\mathrm{d}x + \int_a^b |u_0(x) - v_0(x)|\,\mathrm{d}x \geq 0.$$

Denoting

$$h(t) := \int_{a-M(t-s)}^{b+M(t-s)} |u(t,x) - v(t,x)|\chi'(t)\,\mathrm{d}t\,\mathrm{d}x$$

which is a continuous function, we have therefore

$$\int_0^T h(t)\chi'(t)\,\mathrm{d}t + h(0) \geq 0$$

for all $\chi \in C^1_c((-\infty, s'))$ with $\chi(0) = 1$. Defining ζ_ε as an approximation of the Dirac distribution with mass 1 at s and $\chi(t) = 1 - \int_0^t \zeta_\varepsilon(t)\,\mathrm{d}t$, we deduce by the dominated convergence theorem

$$\int_0^T h(t)\chi'(t)\,\mathrm{d}t + h(0) = -\int_0^T h(t)\zeta_\varepsilon(t)\,\mathrm{d}t + h(0) \xrightarrow[\varepsilon\to 0]{} -h(s) + h(0)$$

which proves that $h(s) \le h(0)$:

$$\int_{a-M(t-s)}^{b+M(t-s)} |u(t,x) - v(t,x)|\,\mathrm{d}t \le \int_a^b |u_0(x) - v_0(x)|\,\mathrm{d}x$$

and since the argument can be repeated for all times $s \in [0,T)$ this proves that $h(s)$ is non-increasing.

Moreover by taking $-a = b \to \infty$ we deduce also that

$$\forall\, t \in [0,T), \quad \|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R})} \le \|u_0 - v_0\|_{L^1(\mathbb{R})}.$$

This a fundamental contraction property, which is the key to solving the Cauchy problem for scalar nonlinear transport equations. Unfortunately no such property (possibly for a more complicated metrics) is known to this day in the case of systems, which is a source of difficulty in this case.

We also deduce, taking $v = 0$ the zero solution

$$\forall\, t \in [0,T), \quad \|u(t,\cdot)\|_{L^1(\mathbb{R})} \le \|u_0\|_{L^1(\mathbb{R})}.$$

Applying the definition of weak solution (implied by that of entropic solutions) with $\varphi(t,x) = \chi(t)\theta(x)$ with χ as above and θ converging to 1 on $\mathbb{R}$, thanks to the previous L^1 bound and the dominated convergence theorem we deduce that

$$\int_0^T \int_{\mathbb{R}} (u(t,x) - v(t,x))\chi'(t)\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} (u_0(x) - v_0(x))\,\mathrm{d}x = 0$$

(be aware that there are no absolute values here), which implies by arguing as before

$$\forall\, t \in [0,T), \quad \int_{\mathbb{R}} (u(t,x) - v(t,x))\,\mathrm{d}x = \int_{\mathbb{R}} (u_0(x) - v_0(x))\,\mathrm{d}x.$$

We can now even deduce *pointwise* properties thanks to these L^1 bounds by a classical simple but elegant argument. Observe that if $u_0 \ge v_0$ almost everywhere on $\mathbb{R}$ then

$$\|u_0 - v_0\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} (u_0(x) - v_0(x))\,\mathrm{d}x.$$

From the bound above we deduce that

$$\int_{\mathbb{R}} (u(t,x) - v(t,x))\,\mathrm{d}x = \int_{\mathbb{R}} (u_0(x) - v_0(x))\,\mathrm{d}x$$
$$= \|u_0 - v_0\|_{L^1(\mathbb{R})} \ge \|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R})}$$

which implies that $u(t,x) \ge v(t,x)$ almost everywhere on $\mathbb{R}$ for $t \in [0,T)$. Observe that taking $v = 0$ the zero solution this proves that if $u_0 \ge 0$ a.e. then the solution remains non-negative a.e. So most of the properties we would expect from the characteristic formula still hold, however now the L^1 norm is not necessarily preserved and can decrease (but not increase).

We can even go further and deduce properties on the total variation of the solution. Let us introduce the space of bounded variations $BV(\mathbb{R})$ as $u_0 \in BV(\mathbb{R})$ iff $u_0(\cdot + h) - u_0(\cdot)$ is integrable for all $h \in \mathbb{R}$ and the following limit exists

$$TV(u_0) = \lim_{h\to 0,\ h\neq 0} \int_{\mathbb{R}} |u_0(x+h) - u_0(x)| \, \mathrm{d}x < +\infty.$$

Then observe that if $u(t,x)$ is a solution then so is $u(t, \cdot + h)$ for any $h \neq 0$. From the previous bounds we deduce that $u(t, \cdot + h) - u(t, \cdot)$ is integrable with

$$\forall\, t \in [0,T), \quad \int_{\mathbb{R}} |u(t,x+h) - u_0(t,x)| \, \mathrm{d}x \leq \int_{\mathbb{R}} |u_0(x+h) - u_0(x)| \, \mathrm{d}x$$

which implies that

$$TV(u(t,\cdot)) \leq TV(u_0).$$

We therefore also have decay of the total variation, which is a stronger first-order regularity property. $\square$

5. The wave equation

In this section we consider the *wave equation*

$$\partial_t^2 u = \Delta_x u = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2 u, \quad u = u(t,x),\ x = (x_1, \ldots, x_n) \tag{5.1}$$

where the total dimension in space-time is $\ell = n+1$. It is common notation to use the shorthand

$$\square := \partial_t^2 - \partial_{x_1}^2 - \cdots - \partial_{x_n}^2$$

for the so-called *d'Alembertian operator* and the wave equation then writes $\square u = 0$. Another common notation is to write $\mathbb{R}^\ell = \mathbb{R}^{1+n}$ and say that the problem has dimension "$1+n$" in order to highlight to role played by the time variable.

This equation is now a *system* of transport equations. In space dimension $n = 1$, define $\mathbf{v} = (v_1, v_2) = (u, \partial_t u + \partial_x u)$, then the equation $\square u = h$ writes

$$\begin{aligned} \partial_t v_1 + \partial_x v_1 &= v_2 \\ \partial_t v_2 - \partial_x v_2 &= h. \end{aligned}$$

In higher space dimension $n \geq 1$, define $D_x := \sqrt{-\Delta_x}$, and $\mathbf{v} = (v_1, v_2) = (u, \partial_t u + iD_x u)$, then the equation $\square u = h$ writes

$$\begin{aligned} \partial_t v_1 + iD_x v_1 &= v_2 \\ \partial_t v_2 - iD_x v_2 &= h. \end{aligned}$$

Hence one sees that they are two, rather than one, set of characteristic trajectories, associated with speed ± 1 (without normalisation of the constants this would be the speed of sound, or light, etc.)

This is one of the most fundamental partial differential equations. It was in fact the first PDE to be studied and its study motivated the development of Fourier analysis. It appeared originally in the context of the small vibrations of a string, and, in the linearisation of the (much more complicated) compressible Euler equations. It is also "included" in the Maxwell equations for the electromagnetic field.

Time independent solutions of (5.1) satisfy the Laplace equation $\Delta u = 0$. Thus, study of the Laplace equation is naturally required in the study of (5.1). The wave equation is the second prototypical example of hyperbolic equations, together with scalar transport equations; it is in fact a prototype of a vectorial (first-order) transport equation, but it is more commonly thought of as a second-order scalar equation. As we discussed in the second chapter, if ellipticity is the property of the absence of characteristic hypersurfaces, hyperbolicity essentially means that "there are as many characteristic hypersurfaces as possible". In particular any smooth hypersurface $\mathcal{S}^{n-1} \subset \mathbb{R}^n$ can be extended to an hypersurface $\tilde{\mathcal{S}}^n \subset \mathbb{R}^{n+1}$ which is characteristic. We shall not attempt here to give a general definition in the way we did for ellipticity but we shall see some important properties of hyperbolic equations on this example.

5.1. Hyperbolicity for second-order linear evolution equations.

5.1.1. *The notion of hyperbolicity.* Let us define our general setting. Consider $\mathcal{U} \subset \mathbb{R}^n$ an open bounded set, and denote $\mathcal{U}_T = \mathcal{U} \times (0,T)$ for any time $T > 0$. Then define the initial-boundary-value (in short "IBV") problem

$$\left\{ \begin{aligned} \partial_t^2 u + \mathfrak{P} u &= f \quad \text{in } \mathcal{U}_T, \\ u &= 0 \quad \text{on } \partial\mathcal{U} \times [0,T], \\ u = u_0, \ \partial_t u &= u_1 \quad \text{on } \{t=0\} \times \mathcal{U}. \end{aligned} \right. \tag{5.2}$$

This corresponds to *Dirichlet conditions* on the (space) boundary. Other boundary conditions could be possible such as the Neumann conditions $\partial_n u = 0$ (where n is the outer normal to the surface $\partial\mathcal{U}$), or the *impedance conditions* $\partial_n u + Z(x)\partial_t u$ with some function $Z(x) \geq 0$... The source term of this problem is $f : \mathcal{U}_T \to \mathbb{R}$, the initial data are $u_0, u_1 : \mathcal{U} \to \mathbb{R}$, and the unknown function is $u : \overline{\mathcal{U}}_T \to \mathbb{R}$.

The operator $\mathfrak{P}$ is defined as

$$\mathfrak{P} u := -\sum_{i,j=1}^n \partial_{x_j} \left(a_{ij}(t,x) \partial_{x_i} u\right) + \sum_{i=1}^n b_i(t,x) \partial_{x_i} u + c(t,x) u \tag{5.3}$$

in *divergence form*, and

$$\mathfrak{P} u := -\sum_{i,j=1}^n a_{ij}(t,x) \partial^2_{x_i x_j} u + \sum_{i=1}^n b_i(t,x) \partial_{x_i} u + c(t,x) u \tag{5.4}$$

in *non-divergence form.* We assume w.l.o.g. that $a_{ij} = a_{ji}$.

Definition 5.1. *The operator $\partial_t^2 + \mathfrak{P}$ is* hyperbolic *at (t,x) if there a constant $\theta(t,x) > 0$ so that*

$$\sum_{i,j=1}^n a_{ij}(t,x) \xi_i \xi_j \geq \theta(t,x) |\xi|^2. \tag{5.5}$$

It is said uniformly hyperbolic *in $\mathcal{U}_T$ if (5.5) holds at every points $(t,x) \in \mathcal{U}_T$ with a uniform constant $\theta > 0$.*

Observe that when $A = (a_{ij}) = \mathrm{Id}$ and $B = (b_i) = 0$, $f = 0$, the operator $\mathfrak{P} = -\Delta_x$ and $\partial_t^2 + \mathfrak{P} = \Box$ corresponds exactly to the wave equation.

EXERCISE 50. *Using the notions of the chapter 2, show that the hypersurface $\mathcal{U} \times \{t = 0\}$ is non-characteristic, which is hint that we expect the Cauchy problem with initial conditions on u and u_t to be well-posed.*

In particular show that, for analytic data, the Cauchy-Kowalevskaya theorem proves the well-posedness for a small time T. Show furthermore that actually these solutions can be extended to an arbitrary T, by studying the radius of convergence of the entire series.

5.2. Energy estimates in the whole Euclidean space. Consider now again the standard wave equation over the whole Euclidean space with source term:

$$\begin{cases} \Box u &= f \quad \text{in } (0,T) \times \mathbb{R}^n, \\ u = u_0, \ \partial_t u &= u_1 \quad \text{on } \{t = 0\} \times \mathbb{R}^n. \end{cases}$$

We will proceed in the exposition as for the Poisson equation. Assume a priori that we have a classical C^2 solution on $\overline{\mathcal{U}}_T$. Assume also a priori that we know that u and its derivatives decay fast enough at infinity in order to have the required integrability in the following estimates (note that this clearly now falls oustide the realm of the Cauchy-Kowalevskaya theorem).

The key (and fundamental) a priori estimate is the following. Multiply the equation by $\partial_t u$ and integrate it over $\overline{\mathcal{U}}_T = [0,T] \times \mathbb{R}^n$:

$$\begin{aligned} 0 = \int_0^T \int_{\mathbb{R}^n} \partial_t u \left(\partial_t^2 u - \Delta_x u\right) \, \mathrm{d}x \, \mathrm{d}t \\ &= \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} \partial_t \left|\partial_t u\right|^2 \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^n} (\partial_t \nabla_x u) \cdot \nabla_x u \, \mathrm{d}x \, \mathrm{d}t \\ &= \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} \partial_t \left|\partial_t u\right|^2 \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} \partial_t \left|\nabla_x u\right|^2 \, \mathrm{d}x \, \mathrm{d}t \\ &= \left[\frac{1}{2} \int_{\mathbb{R}^n} \left|\partial_t u\right|^2 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^n} \left|\nabla_x u\right|^2 \, \mathrm{d}x \right]_0^T \\ &= \left[\frac{1}{2} \int_{\mathbb{R}^n} \left|\partial_t u(T,x)\right|^2 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^n} \left|\nabla_x u(T,x)\right|^2 \, \mathrm{d}x \right] \\ &\qquad - \left[\frac{1}{2} \int_{\mathbb{R}^n} \left|\partial_t u(0,x)\right|^2 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^n} \left|\nabla_x u(0,x)\right|^2 \, \mathrm{d}x \right]. \end{aligned}$$

We hence deduce the conservation along time of the following *energy* (hence the name "energy method"):

$$\mathcal{E}(t) := \left[\frac{1}{2} \int_{\mathbb{R}^n} \left|\partial_t u(t,x)\right|^2 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^n} \left|\nabla_x u(t,x)\right|^2 \, \mathrm{d}x \right]$$

and this gives

$$\forall\, t \geq 0, \quad \|\partial_t u(t,\cdot)\|^2_{L^2_x(\mathbb{R}^n)} + \|u(t,\cdot)\|^2_{\dot{H}^1_x(\mathbb{R}^n)} \leq \|u_0\|^2_{\dot{H}^1(\mathbb{R}^n)} + \|u_1\|^2_{L^2(\mathbb{R}^n)}.$$

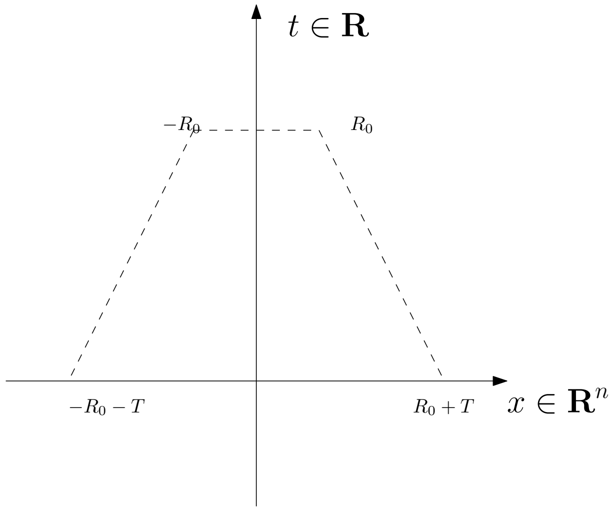

FIGURE 5.1. Trapezium C for the local energy estimate.

EXERCISE 51. *Applying this to a difference of two classical solutions with same initial data and decay at infinity, prove the uniqueness of such solutions.*

5.3. Local energy estimates. The fundamental a priori estimate will now be made much more interesting by making it local and uncovering the *cone of dependence*. This is reminiscent of the uniqueness estimate we proved in the Kružkov Theorem. The idea is now to apply the energy estimate to the trapezium

$$\mathcal{C} := \bigcup_{t \in [0,T]} \{t\} \times B(x_0, R_0 + T - t)$$

for some given base point x_0 and radius $R_0 > 0$ (see Figure above).

Let us define the *local* energy

$$\mathcal{E}(t) := \int_{B(x_0,R_0+T-t)} \left(\frac{1}{2} \left|\partial_t u(t,x)\right|^2 + \frac{1}{2} \left|\nabla_x u(t,x)\right|^2 \right) \mathrm{d}x$$

and differentiate it in time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = \int_{B(x_0,R_0+T-t)} \Big(\partial_t^2 u(t,x) \partial_t u(t,x)\, \mathrm{d}x + (\partial_t \nabla_x u(t,x)) \cdot \nabla_x u(t,x) \Big)\, \mathrm{d}x$$
$$- \int_{\partial B(x_0,R_0+T-t)} \left(\frac{1}{2} \left|\partial_t u(t,x)\right|^2 + \frac{1}{2} \left|\nabla_x u(t,x)\right|^2 \right) \mathrm{d}S$$

where the second integral is a surface integral.

EXERCISE 52. *Check this calculation by computing the derivative of* $t \mapsto \int_{B(x_0,R_0+T-t)} F \,\mathrm{d}x$ *for some* F *independent of* t.

We now perform an integration by parts in the second term in the RHS:

$$\int_{B(x_0,R_0+T-t)} (\partial_t \nabla_x u(t,x)) \cdot \nabla_x u(t,x)\, \mathrm{d}x$$

$$= - \int_{B(x_0,R_0+T-t)} \partial_t u(t,x) \Delta_x u(t,x) \, \mathrm{d}x + \int_{\partial B(x_0,R_0+T-t)} \partial_t u(t,x) \frac{\partial u}{\partial n}(t,x) \, \mathrm{d}S$$

where $\partial u / \partial n$ denotes the normal (outer) derivative of $\partial B(x_0, R_0 + T - t)$.

We deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = &\int_{B(x_0,R_0+T-t)} \partial_t u(t,x) \Big(\partial_t^2 u(t,x) - \Delta_x u(t,x) \Big) \, \mathrm{d}x \\ &+ \int_{\partial B(x_0,R_0+T-t)} \left(\partial_t u(t,x) \frac{\partial u}{\partial n}(t,x) - \frac{1}{2} |\partial_t u(t,x)|^2 - \frac{1}{2} |\nabla_x u(t,x)|^2 \right) \mathrm{d}S.\end{aligned}$$

The first term is zero thanks to the PDE, and in the second term observe that

$$\partial_t u(t,x) \frac{\partial u}{\partial n}(t,x) \le \frac{1}{2} |\partial_t u(t,x)|^2 + \frac{1}{2} \left| \frac{\partial u}{\partial n}(t,x) \right|^2 \le \frac{1}{2} |\partial_t u(t,x)|^2 + \frac{1}{2} |\nabla_x u(t,x)|^2$$

which shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \le 0 \quad \Longrightarrow \quad 0 \le \mathcal{E}(t) \le \mathcal{E}(0).$$

This implies the fundamental property of the *cone of dependency*:

PROPOSITION 5.2. *For a classical solution, if $u = \partial_t u = 0$ on $B(x_0, R_0 + T)$ at time $t = 0$, then $u = 0$ on the cone $\mathcal{C}$ defined above.*

REMARK 5.3. *Observe that this fundamental property is not seen in the analytic theory. For the wave equation it can be deduced from the explicit solutions but it is* much *simpler to obtain it from energy estimates, and much more robust when dealing with more general second-order hyperbolic equations or with nonlinear equations.*

Let us now discuss a refinement of this a priori estimate that illustrates a key principle, already encountered with parabolic equations: if a Lyapunov function is found for a PDE, say which decays, it gives *two* bounds and not just *one*, by considering the signed time-derivative of this quantity. Here

$$\begin{aligned}\mathcal{D}(t) &:= -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \\ &= \int_{\partial B(x_0,R_0+T-t)} \left(\frac{1}{2} |\partial_t u(t,x)|^2 + \frac{1}{2} |\nabla_x u(t,x)|^2 - \partial_t u(t,x) \frac{\partial u}{\partial n}(t,x) \right) \mathrm{d}S\end{aligned}$$

has to be time-integrable.

Let us now decompose $n_t := (1, 0, \dots, 0) \in \mathbb{R}^{1+n}$ into n normal vector to the cone and m_V some vector field tangential to the cone. Let us also denote ∇_x^V the x-gradient tangential to the cone, i.e. along ∂B_{R+T-t}. Then

$$\begin{cases} \partial_t = \dfrac{\partial}{\partial n} + \dfrac{\partial}{\partial m_V} \\ |\nabla_x u(t,x)|^2 = \left|\nabla_x^V u(t,x)\right|^2 + \left|\dfrac{\partial u}{\partial n}(t,x)\right|^2 \\ |\partial_t u(t,x)|^2 = \left|\dfrac{\partial u}{\partial n}(t,x)\right|^2 + \left|\dfrac{\partial u}{\partial m_V}(t,x)\right|^2 + 2\dfrac{\partial u}{\partial n}(t,x)\dfrac{\partial u}{\partial m_V}(t,x) \\ \partial_t u(t,x)\dfrac{\partial u}{\partial n}(t,x) = \left|\dfrac{\partial u}{\partial n}(t,x)\right|^2 + \dfrac{\partial u}{\partial n}(t,x)\dfrac{\partial u}{\partial m_V}(t,x) \end{cases}$$

which shows that

$$\mathcal{D}(t) = \int_{\partial B(x_0, R_0+T-t)} \left(\frac{1}{2}\left|\frac{\partial u}{\partial m_V}(t,x)\right|^2 + \frac{1}{2}\left|\nabla_x^V u(t,x)\right|^2 \right) \mathrm{d}S.$$

Remark 5.4. *Here are some additional comments on this key a priori estimate. It can in fact be performed with other non-characteristic surfaces. More generally one can show that a characteristic hypersurface of $\Box$ can be locally represented as level sets $g = cst$ of solutions g to the so-called* eikonal equation

$$-(\partial_t g)^2 + |\nabla_x g|^2 = 0.$$

Correspondingly a non-characteristic hypersurface can be locally represented as a level set of a g with $-(\partial_t g)^2 + |\nabla_x g|^2 \neq 0$. The hypersurface is called spacelike *if $-(\partial_t g)^2 + |\nabla_x g|^2 < 0$ and* timelike *otherwise. On the one hand, the local energy estimate can be generalized to spacelike non-characteristic hypersurface, with a boundary of the domain of dependence being a characteristic hypersurface and the associated boundary term in the estimates being non-positive. On the other hand, the local energy estimate for a timelike non-characteristic hypersurface gives nothing as the boundary term has no sign. And it turns out that in this case the Cauchy problem is ill-posed (for $n \geq 2$).*