Homework #2
(Solutions)

1. Find the arclength of the curve $y = \frac{2\sqrt{3}}{9}(3x^2+1)^{3/2}$ from $x=-1$ to $x=2$

Solution: $y' = \frac{\sqrt{3}}{3}(3x^2+1)^{1/2} \cdot 6x$

$\Rightarrow (y')^2 = \cancel{\tfrac{4}{3}}\ 12x^2(3x^2+1) = 36x^4+12x^2$

$\Rightarrow (y')^2+1 = 36x^4+12x^2+1$

Using the formula: Arclength $= \int_a^b \sqrt{1+(y')^2}\,dx$, we have:

$$\int_{-1}^{2} \sqrt{36x^4+12x^2+1}\,dx$$

If $w=6x^2$, then $36x^4+12x^2+1 \rightarrow w^2+2w+1$, which factors as $(w+1)^2$. Substituting back in, this becomes $(6x^2+1)^2$. Hence the integral becomes:

$$\int_{-1}^{2} \sqrt{(6x^2+1)^2}\,dx = \int_{-1}^{2} 6x^2+1\,dx = 2x^3+x\Big|_{-1}^{2}$$

$= 18-(-3) = \boxed{21}$.

2. Assume the position of a particle at each time t (measured in terms of distance to the origin) is given by $s(t) = \frac{1}{12}e^{t} + 3e^{-t}$. How far has the particle traveled during the time interval $t = \ln(2)$ to $t = \ln(4)$?

Solution: First, we see when the particle might change direction in the interval. This happens when $v(t) = 0$:

$$v(t) = s'(t) = \frac{1}{12}e^{t}\cancel{t} - 3e^{-t} = \frac{e^{t}}{12} - \frac{3}{e^{t}} =$$

$$= \frac{e^{2t} - 36}{12e^{t}}.$$ Setting the numerator $= 0$ we obtain: $e^{2t} - 36 = 0 \Rightarrow e^{2t} = 36 \Rightarrow$

$$2t = \ln(36) \Rightarrow t = \tfrac{1}{2}\ln(36) = \ln(36^{1/2}) = \ln(6).$$

Hence $v(t) = 0$ at $t = \ln(6)$. But this is outside our time interval $t = \ln(2)$ to $t = \ln(4)$. Therefore, in our interval the particle moves in a single direction, and we can simply find

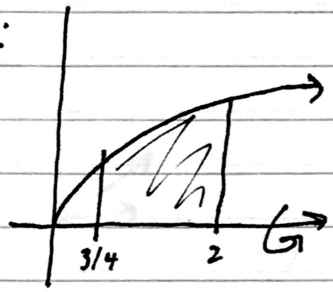
$$\left| s(\ln(4)) - s(\ln(2)) \right| = \left| \left(\frac{1}{12}e^{\ln(4)} + 3e^{-\ln(4)} \right) - \left(\frac{1}{12}e^{\ln(2)} + 3e^{-\ln(2)} \right) \right|$$

$$= \left| \left(\frac{4}{12} + \frac{3}{4} \right) - \left(\frac{2}{12} + \frac{3}{2} \right) \right| = \left| \frac{2}{12} - \frac{3}{4} \right| = \left| \frac{1}{6} - \frac{3}{4} \right| =$$

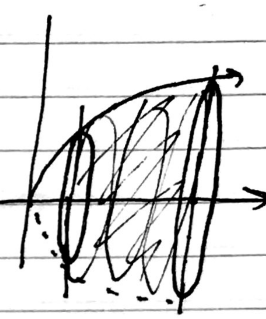
$$= \left| \frac{-14}{24} \right| = \boxed{7/12}$$

3. Compute the area of the surface obtained by rotating $y = \sqrt{x}$, $\frac{3}{4} \leq x \leq 2$ about the x-axis.

<u>Solution</u> :



$\leadsto$



We have the formula: $S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx$

$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow 1 + (f'(x))^2 = 1 + \frac{1}{4x}$

Hence $S = \int_{3/4}^{2} 2\pi \sqrt{x} \sqrt{1 + \frac{1}{4x}}\, dx$

$= \int_{3/4}^{2} 2\pi \sqrt{x\left(1 + \frac{1}{4x}\right)}\, dx = \int_{3/4}^{2} 2\pi \sqrt{x + \frac{1}{4}}\, dx.$

Letting $u = x + \frac{1}{4} \Rightarrow du = dx$, so we have:

$\int 2\pi \sqrt{u}\, du = 2\pi \cdot \frac{2}{3} u^{3/2} = \frac{4\pi}{3} \left(x + \frac{1}{4}\right)^{3/2} \Big|_{3/4}^{2}$

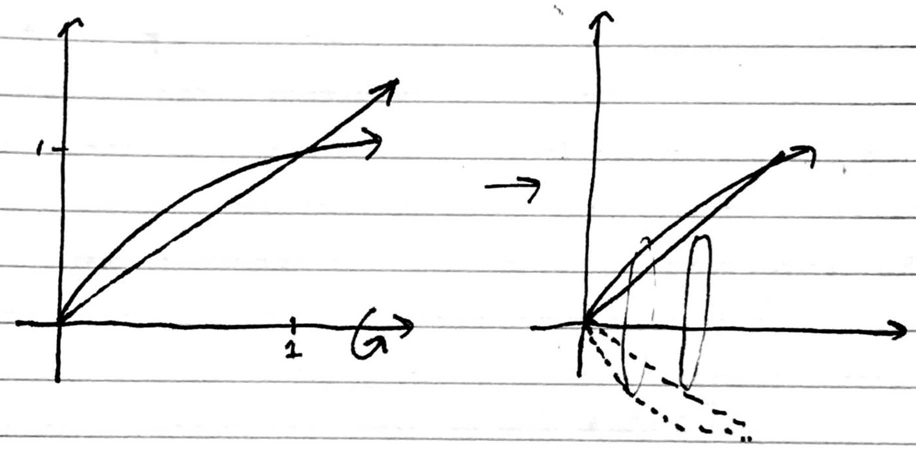
$= \frac{4\pi}{3} \left(\frac{9}{4}\right)^{3/2} - \frac{4\pi}{3} (1)^{3/2} = \frac{4\pi}{3} \left(\frac{3}{2}\right)^3 - \frac{4\pi}{3}$

$= \frac{4\pi}{3} \cdot \frac{27}{8} - \frac{4\pi}{3} = \frac{108\pi}{24} - \frac{4\pi}{3} \Rightarrow$

$\cancel{= \boxed{\frac{92\pi}{3}} = \boxed{\frac{65\pi}{42}}} \quad = \boxed{\frac{19\pi}{6}}$

4. Region bounded by $y=\sqrt{x}$, $y=x$, between $x=0$ and $x=1$, rotated about the x-axis. What is the surface area?

<u>Solution</u>:



<u>Surface Area of Exterior</u>: Using $S=\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$, with $f(x)=\sqrt{x}$, we obtain:

$$S=\int_0^1 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx = \int_0^1 2\pi\sqrt{x+\frac{1}{4}}\,dx$$

As in problem 3, this equals:

$$\frac{4}{3}\pi\left(x+\frac{1}{4}\right)^{3/2}\Big|_0^1 = \frac{4}{3}\pi\left(\frac{5}{4}\right)^{3/2}-\frac{4}{3}\pi\left(\frac{1}{4}\right)^{3/2}$$

$$= \pi(5\sqrt{5}-1)/6.$$

<u>Surface Area of Interior</u>: Using $S=\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$ with $f(x)=x$, we obtain:

$$S=\int_0^1 2\pi x\sqrt{2}\,dx = \pi\sqrt{2}\,x^2\Big|_0^1 = \pi\sqrt{2}$$

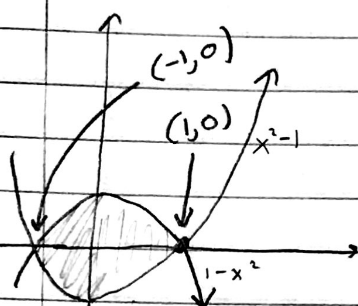
Therefore, the total surface area is:

$$\frac{\pi(5\sqrt{5}-1)}{6}+\pi\sqrt{2} = \boxed{\frac{\pi}{6}\left(5\sqrt{5}-1+6\sqrt{2}\right)}$$

5. Find the center of mass of the region bounded by $y = 1 - x^2$ and $y = x^2 - 1$, with $\delta(x,y) = -x^2 + 2x + 3$.

Solution:

$\frac{112}{15}$



$$\bar{x} = \frac{1}{M}\int_a^b x\,\delta(x)\big(f(x) - g(x)\big)\,dx$$

$$\bar{y} = \frac{1}{M}\int_a^b \frac{\delta(x)}{2}\big((f(x))^2 - (g(x))^2\big)\,dx$$

$$M = \int_a^b \delta(x)\big(f(x) - g(x)\big)\,dx$$

Using these formulas, we have:

$$M = \int_{-1}^{1} (-x^2+2x+3)\big((1-x^2)-(x^2-1)\big)\,dx =$$

$$= \int_{-1}^{1} (-x^2+2x+3)(-2x^2+2)\,dx = \int_{-1}^{1} 2x^4 - 4x^3 - 6x^2 - 2x^2 + 4x + 6\,dx$$

$$= \int_{-1}^{1} 2x^4 - 4x^3 - 8x^2 + 4x + 6\,dx = \frac{2}{5}x^5 - x^4 - \frac{8}{3}x^3 + 2x^2 + 6x\Big|_{-1}^{1}$$

$$= \left(\frac{2}{5} - 1 - \frac{8}{3} + 2 + 6\right) - \left(\frac{-2}{5} - 1 + \frac{8}{3} + 2 - 6\right) = 112/15$$

$$\bar{x} = \frac{15}{112}\int_{-1}^{1} x(-x^2+2x+3)\big((1-x^2)-(x^2-1)\big)\,dx$$

$$= \frac{15}{112}\int_{-1}^{1} 2x^5 - 4x^4 - 8x^3 + 4x^2 + 6x\,dx =$$

$$= \frac{15}{112}\left(\frac{1}{3}x^6 - \frac{4}{5}x^5 - 2x^4 + \frac{4}{3}x^3 + 3x^2\right)\Big|_{-1}^{1} = \frac{15}{112}\left(\frac{16}{15}\right) = \frac{1}{7}$$

$$\bar{y} = \frac{15}{112}\int_{-1}^{1} \frac{1}{2}(-x^2+2x+3)\big((1-x^2)^2 - (x^2-1)^2\big)\,dx$$

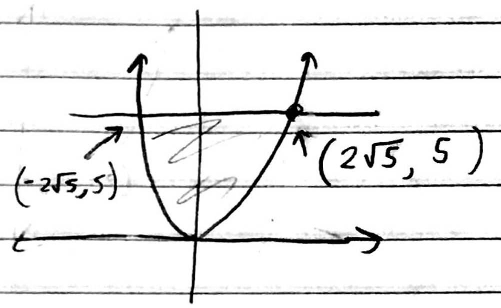
$$= \frac{15}{112}(0) = 0$$

Hence the center of mass is

$$\boxed{(\bar{x}, \bar{y}) = (1/7, 0)}.$$

6. Find the centroid of the region bounded by $y = x^2/4$ and $y = 5$

Solution:



The centroid is simply the center of mass when density is constant. Hence the formulas are:

$$\bar{x} = \frac{M_y}{M} = \frac{\delta \int_a^b x(f(x) - g(x))\,dx}{\delta \int_a^b f(x) - g(x)\,dx} = \frac{\int_a^b x(f(x) - g(x))\,dx}{\int_a^b f(x) - g(x)\,dx}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\delta \int_a^b \frac{1}{2}((f(x))^2 - (g(x))^2)\,dx}{\delta \int_a^b f(x) - g(x)\,dx} = \frac{\int_a^b \frac{1}{2}(f(x)^2 - g(x)^2)\,dx}{\int_a^b f(x) - g(x)\,dx}$$

We have $\int_a^b f(x) - g(x)\,dx = \int_{-2\sqrt{5}}^{2\sqrt{5}} 5 - \frac{x^2}{4}\,dx = 5x - \frac{1}{12}x^3 \Big|_{-2\sqrt{5}}^{2\sqrt{5}}$
$= 40\sqrt{5}/3$

$\int_a^b x(f(x) - g(x))\,dx = \int_{-2\sqrt{5}}^{2\sqrt{5}} x\left(5 - \frac{x^2}{4}\right)dx = \int_{-2\sqrt{5}}^{2\sqrt{5}} 5x - \frac{x^3}{4}\,dx = \frac{5}{2}x^2 - \frac{1}{16}x^4 \Big|_{-2\sqrt{5}}^{2\sqrt{5}}$
$= 0$

And $\int_a^b \frac{1}{2}(f(x)^2 - g(x)^2)\,dx = \int_{-2\sqrt{5}}^{2\sqrt{5}} \frac{1}{2}\left(5^2 - \left(\frac{x^2}{4}\right)^2\right)dx = \int_{-2\sqrt{5}}^{2\sqrt{5}} \frac{25}{2} - \frac{x^4}{32}\,dx$
$= \frac{25}{2}x - \frac{1}{160}x^5 \Big|_{-2\sqrt{5}}^{2\sqrt{5}} = 40\sqrt{5}.$

Hence $\bar{x} = \frac{0}{(40\sqrt{5}/3)} = 0$

$\bar{y} = \frac{40\sqrt{5}}{(40\sqrt{5}/3)} = 3$

$\Rightarrow (\bar{x}, \bar{y}) = \boxed{(0, 3)}$.

(or, $\bar{x}$ by observation)