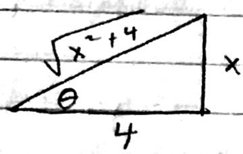
Homework # 4
Solutions

1. a) $\int \frac{dx}{x^2\sqrt{x^2+4}}$



$\cot\theta = \frac{4}{x} \Rightarrow \frac{1}{4}\cot\theta = \frac{1}{x} \Rightarrow \frac{1}{16}\cot^2\theta = \frac{1}{x^2}$

$\cos\theta = \frac{4}{\sqrt{x^2+4}} \Rightarrow \frac{1}{4}\cos\theta = \frac{1}{\sqrt{x^2+4}}$

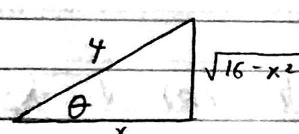
$\tan\theta = \frac{x}{4} \Rightarrow 4\tan\theta = x \Rightarrow 4\sec^2\theta\, d\theta = dx$

$\rightsquigarrow \int \frac{1}{16}\cot^2\theta\cos\theta\sec^2\theta\, d\theta = \frac{1}{16}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$

$u = \sin\theta,\ du = \cos\theta\, d\theta \rightsquigarrow \frac{1}{16}\int \frac{1}{u^2}\, du = -\frac{1}{16}\cdot\frac{1}{u} + C$

$= -\frac{1}{16}\csc\theta + C = -\frac{1}{16}\cdot\frac{\sqrt{x^2+4}}{4} = \boxed{-\sqrt{x^2+4}/64}$

b) $\int_0^{2\sqrt{2}} \frac{x^2}{(16-x^2)^{3/2}}\, dx$



$4\cos\theta = x \Rightarrow 16\cos^2\theta = x^2$

$\frac{1}{4}\csc\theta = \frac{1}{\sqrt{16-x^2}} \Rightarrow \frac{1}{64}\csc^3\theta = \frac{1}{(16-x^2)^{3/2}}$

$4\cos\theta = x \Rightarrow -4\sin\theta\, d\theta = dx$

$\rightsquigarrow \int 16\cos^2\theta\cdot\frac{1}{64}\csc^3\theta\cdot -4\sin\theta\, d\theta =$

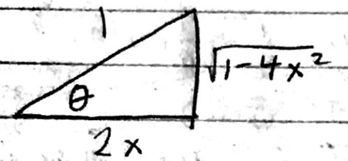
$= -\int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta = -\int \cot^2\theta\, d\theta = -\int \csc^2\theta - 1\, d\theta = \cot\theta + \theta$

$= \frac{x}{\sqrt{16-x^2}} + \arccos\left(\frac{x}{4}\right)\Big|_0^{2\sqrt{2}} = \frac{-2\sqrt{2}}{\sqrt{16-8}} + \arccos\left(\frac{\sqrt{2}}{2}\right) - (0 + \pi/2)$

$= \frac{2\sqrt{2}}{\sqrt{8}} + \frac{\pi}{4} - \frac{\pi}{2} =$

$= \boxed{1 - \frac{\pi}{4}} =$

c) $\int_0^{1/2} \sqrt{1-4x^2}\, dx$



$\sin(\theta) = \sqrt{1-4x^2}$

$\cos(\theta) = 2x \Rightarrow \frac{1}{2}\cos(\theta) = x \Rightarrow -\frac{1}{2}\sin(\theta)d\theta = dx$

$\rightsquigarrow \int -\frac{1}{2}\sin^2\theta\, d\theta = \int -\frac{1}{4}\left(1-\cos(2\theta)\right) d\theta$

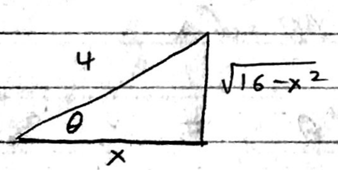
$= -\frac{1}{4}\theta + \int \frac{1}{4}\cos(2\theta)\, d\theta = -\frac{1}{4}\theta + \frac{1}{8}\sin(2\theta)$

$= -\frac{1}{4}\theta + \frac{1}{4}\sin\theta\cos\theta = -\frac{1}{4}\arccos(2x) + \frac{x}{2}\sqrt{1-4x^2}\Big|_0^{1/2}$

$= \left(-\frac{1}{4}\arccos(1) + 0\right) - \left(-\frac{1}{4}\arccos(0) + 0\right)$

$= (0+0) - \left((-\pi/8) + 0\right) = \boxed{\pi/8}$

d) $\int_0^2 \sqrt{16-x^2}\, dx$



$\sin(\theta) = \frac{\sqrt{16-x^2}}{4} \Rightarrow 4\sin(\theta) = \sqrt{16-x^2}$

$\cos(\theta) = \frac{x}{4} \Rightarrow 4\cos(\theta) = x \Rightarrow -4\sin(\theta)d\theta = dx$

$\rightsquigarrow \int -16\sin^2(\theta)\, d\theta = \int -8(1-\cos(2\theta))\, d\theta$

$= -8\theta + \int 8\cos(2\theta)d\theta = -8\theta + 4\sin(2\theta) = -8\theta + 8\sin\theta\cos\theta$

$= -8\arccos\left(\frac{x}{4}\right) + \frac{x}{2}\sqrt{16-x^2}\Big|_0^2 = \cancel{-8\arccos(\tfrac{1}{2})}$

$= \left(-8\arccos\left(\frac{1}{2}\right) + \sqrt{12}\right) - \left(-8\arccos(0) + 0\right)$

$= -8\left(\frac{\pi}{3}\right) + 2\sqrt{3} + 8\left(\frac{\pi}{2}\right) = 4\pi - \frac{8\pi}{3} + 2\sqrt{3} =$

$= \boxed{\frac{4\pi}{3} + 2\sqrt{3}}$

2) $f(x) = \frac{\sin(x)}{x}$ over $0 \le x \le \pi$

a) Trapezoidal rule with four sub-intervals

Solution: $\Delta x = \frac{b-a}{n} = \frac{\pi}{4} \rightsquigarrow x_0 = 0, x_1 = \frac{\pi}{4}, x_2 = \frac{\pi}{2}, x_3 = \frac{3\pi}{4}, x_4 = \pi$

$$T = \frac{1}{2}\left(\frac{\pi}{4}\right)\left[\frac{\sin(0)}{0} + 2\cdot\frac{\sin(\pi/4)}{\pi/4} + 2\cdot\frac{\sin(\pi/2)}{\pi/2} + 2\cdot\frac{\sin(3\pi/4)}{3\pi/4} + \frac{\sin(\pi)}{\pi}\right]$$

$\hookrightarrow$ replace w/ $\lim_{x\to 0^+} \frac{\sin(x)}{x} = 1$

$$\text{So } T = \frac{\pi}{8}\left(1 + \frac{4\sqrt{2}}{\pi} + \frac{4}{\pi} + \frac{4\sqrt{2}}{3\pi} + 0\right)$$

$$= \frac{\pi}{8}\left(\frac{3\pi + 12\sqrt{2} + 12 + 4\sqrt{2}}{3\pi}\right) = \boxed{\frac{1}{24}\left(3\pi + 16\sqrt{2} + 12\right)}$$

b) Simpson's rule with four sub-intervals:

Solution: As above, $\Delta x = \pi/4$, $x_0 = 0$, $x_1 = \pi/4$, $x_2 = \pi/2$, $x_3 = 3\pi/4$, $x_4 = \pi$

limit!

$$S = \frac{1}{3}\left(\frac{\pi}{4}\right)\left[\frac{\sin(0)}{0} + 4\cdot\frac{\sin(\pi/4)}{\pi/4} + 2\cdot\frac{\sin(\pi/2)}{\pi/2} + 4\cdot\frac{\sin(3\pi/4)}{3\pi/4} + \frac{\sin(\pi)}{\pi}\right]$$

$$= \frac{\pi}{12}\left[1 + \frac{8\sqrt{2}}{\pi} + \frac{4}{\pi} + \frac{8\sqrt{2}}{3\pi} + 0\right]$$

$$= \frac{\pi}{12}\left(\frac{3\pi + 24\sqrt{2} + 12 + 8\sqrt{2}}{3\pi}\right) = \boxed{\frac{1}{36}\left(3\pi + 32\sqrt{2} + 12\right)}$$

3. Find the area enclosed by $y = \frac{1}{x+1}$, $y = \frac{1}{x+2}$ on $[0, \infty)$

<u>Solution</u>: $A = \int_0^\infty \frac{1}{x+1} - \frac{1}{x+2} \, dx$

$$= \ln(x+1) - \ln(x+2) \Big|_0^\infty = \ln\left(\frac{x+1}{x+2}\right)\Big|_0^\infty$$

$$= \left[\lim_{x \to \infty} \left(\ln\left(\frac{x+1}{x+2}\right)\right)\right] - \ln(1/2)$$

$$= \overset{0}{\cancel{\ln(1)}} - \ln(1/2) = \boxed{-\ln(1/2)}$$

4. Compute if converge, or justify why diverge:

(a) $\int_1^\infty \frac{e^{2x}}{x^2}\,dx$

$\boxed{\text{Diverges}}$

Proof: We know that $e^{2x} > x^2$ (for x large enough). Hence $1 < \frac{e^{2x}}{x^2}$. Take $g(x) = 1$ in the direct comparison test. We just showed that $g(x) < \frac{e^{2x}}{x^2}$. Moreover, $\int_1^\infty 1\,dx$ diverges. Hence $\int_1^\infty e^{2x}/x^2\,dx$ diverges as well. ■

b) $\int_1^\infty x^2 e^{-2x}\,dx$

$\boxed{\text{Converges}}$

Solution: IBP – $u = x^2$, $du = 2x\,dx$, $v = -\frac{1}{2}e^{-2x}$, $dv = e^{-2x}$

$$\rightsquigarrow -\frac{x^2}{2}e^{-2x} + \int x e^{-2x}\,dx$$

$u = x$, $du = dx$, $v = -\frac{1}{2}e^{-2x}$, $dv = e^{-2x}dx$

$$\rightsquigarrow -\frac{x^2}{2}e^{-2x} + \left(-\frac{x}{2}e^{-2x} + \int \frac{1}{2}e^{-2x}\,dx\right)$$

$$= -\frac{x^2}{2}e^{-2x} - \frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\Big|_1^\infty$$

$$= \left(\lim_{x\to\infty} -\cancel{\frac{x^2}{2e^{2x}}}^{0} - \cancel{\frac{x}{2e^{2x}}}^{0} - \cancel{\frac{1}{4}e^{-2x}}^{0}\right) - \left(\frac{-1}{2e^2} - \frac{1}{2e^2} - \frac{1}{4e^2}\right)$$

$$= \boxed{5/4e^2}$$

c) $\int_0^1 \frac{dx}{(2x-1)^{1/3}}$

Solution: Discontinuity at $x = 1/2$. Hence

$$\leadsto \int_0^{1/2} \frac{dx}{(2x-1)^{1/3}} + \int_{1/2}^{1} \frac{dx}{(2x-1)^{1/3}}$$

$$= \frac{1}{2}\cdot\frac{3}{2}(2x-1)^{2/3}\Big|_0^{1/2} + \frac{1}{2}\cdot\frac{3}{2}(2x-1)^{2/3}\Big|_{1/2}^{1}$$

$$= \left[\left(\lim_{x\to 1/2^-} \frac{3}{4}(2x-1)^{2/3}\right) - \frac{3}{4}\right] + \left[\left(\frac{3}{4} - \lim_{x\to 1/2^+}\left(\frac{3}{4}(2x-1)^{2/3}\right)\right)\right]$$

$$= \left[0 - \frac{3}{4}\right] + \left[\frac{3}{4} - 0\right] = \boxed{0}$$

d) $\int_0^{\infty} \frac{3x}{x^2+4}\,dx$

$\boxed{\text{Diverges}}$

Proof: Notice that as $x \to \infty$, $\frac{3x}{x^2+4}$ behaves like $\frac{3x}{x^2} = \frac{3}{x}$, so we suspect that $g(x) = \frac{1}{x}$ will work with the limit comparison test. Indeed:

$$\lim_{x\to\infty} \frac{\left(\frac{3x}{x^2+4}\right)}{(1/x)} = \lim_{x\to\infty} \frac{3x^2}{x^2+4} = 3.$$

Since this limit is finite and nonzero, the limit comparison test applies, and as $\int_1^{\infty} \frac{1}{x}$ diverges, $\int_0^{\infty} 3x/(x^2+4)\,dx$ diverges as well ∎.

4 c) $\int_0^1 \frac{4x}{(x+1)(x^2+1)}\,dx$

Solution: Partial fractions –

$$\frac{4x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \Leftrightarrow$$

$$\frac{4x}{(x+1)(x^2+1)} = \frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)} \Leftrightarrow$$

$$4x = A(x^2+1)+(Bx+C)(x+1) \Leftrightarrow$$

$$4x = Ax^2 + A + Bx^2 + Bx + Cx + C \Leftrightarrow$$

$$4x = (A+B)x^2 + (B+C)x + (A+C) \Leftrightarrow$$

$$A+B=0,\ B+C=4,\ A+C=0 \Leftrightarrow$$

$$A=-2,\ B=2,\ C=2$$

$$\rightsquigarrow -2\int \frac{1}{x+1}\,dx + 2\int \frac{x+1}{x^2+1}\,dx$$

$$= -2\ln|x+1| + 2\int \frac{x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1}\,dx$$

$$= -2\ln|x+1| + \ln|x^2+1| + 2\arctan(x)\Big|_0^1$$

$$= -2\ln(2) + \ln(2) + 2\cdot\frac{\pi}{4} - (0+0+0)$$

$$= \boxed{-\ln(2) + \frac{\pi}{2}}$$

f) $\int_4^\infty \frac{dx}{x^2-6x+10}$

$\boxed{\text{converges}}$

<u>Solution</u>: Complete square –

$x^2-6x+10=y \to x^2-6x=y-10 \to$

$x^2-6x+9=y-10+9 \to (x-3)^2=y-1$

$\to y=(x-3)^2+1$.

$\rightsquigarrow \int_4^\infty \frac{dx}{(x-3)^2+1} = \arctan(x-3)\Big|_4^\infty$

$= \left(\lim_{x\to\infty} \arctan(x-3)\right) - \arctan(1)$

$= \pi/2 - \pi/4 = \boxed{\pi/4}$

g) $\int_0^{\pi/2} \tan(\theta)\,d\theta$

<u>Solution</u>: Discontinuous at $\pi/2$ $\left(\frac{\sin(\pi/2)}{\cos(\pi/2)} = 1/0\right)$

$\rightsquigarrow \lim_{b\to\pi/2^-} \int_0^b \tan(\theta)\,d\theta$

$= \lim_{b\to\pi/2^-} \left(-\ln|\cos(\theta)|\Big|_0^b\right)$

$= \left[\lim_{b\to\pi/2^-} \left(-\ln|\cos(b)|\right)\right] - 0$

$= -(-\infty) = \boxed{\infty}$

$\boxed{\text{diverges}}$