1. (a) Let $x = 3\tan u$

$x^2+9 = 9\sec^2 u.$

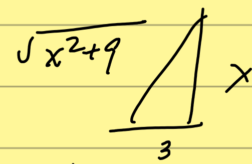
$dx = 3\sec^2 u\, du.$

$$\int \frac{2dx}{x^2\sqrt{x^2+9}} = \int \frac{2\cdot 3\sec^2 u\, du}{9\tan^2 u \quad 3\sec u.}$$

$$= \frac{2}{9}\int \frac{\cos u}{\sin^2 u}\, du$$

$$= \frac{2}{9}\int \frac{d\sin u}{\sin^2 u}$$

$$= \frac{2}{9} \quad -(\sin u)^{-1} + C$$



Now $x = 3\tan u.$

$\tan u = \frac{x}{3}.$ $\Rightarrow$ $\sin u = \frac{x}{\sqrt{x^2+9}}$

$\Rightarrow$ Final answer: $-\frac{2}{9}\,\frac{\sqrt{x^2+9}}{x} + C$

(b) Let $x = 4\sin u.$ $dx = 4\cos u\, du.$

$16 - x^2 = 16\cos^2 u$

$x \in [0, 2\sqrt{2}] \Rightarrow u \in [0, \frac{\pi}{4}].$

the integral becomes $\int_0^{\frac{\pi}{4}} \frac{16\sin^2 u}{4^3\cdot\cos^3 u}\, 4\cos u\, du.$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin^2 u}{\cos^2 u}\, du$$

Recall that $(\tan\theta)' = \sec^2\theta$

$= \tan^2\theta + 1.$

$$= \int_0^{\frac{\pi}{4}} (\sec^2 u - 1)\, du$$

$$= (\tan u - u)\Big|_0^{\frac{\pi}{4}} = 1 - \frac{\pi}{4}$$

(c) Let $x = 4\sin u$ $\quad dx = 4\cos u\, du.$

$x \in [1, 3] \Rightarrow u = [\arcsin\frac{1}{4}, \arcsin\frac{3}{4}].$

$$\int \sqrt{16 - x^2}\, dx = \int_{\arcsin\frac{1}{4}}^{\arcsin\frac{3}{4}} 4\cos u \cdot 4\cos u\, du$$

$$= \int 16\cos^2 u\, du = 16 \int \frac{1+\cos 2u}{2}\, du$$

$$= 16 \left(\frac{u}{2} - \frac{\sin 2u}{4} \right) \Bigg|_{\arcsin\frac{1}{4}}^{\arcsin\frac{3}{4}}$$

When $\sin u = \frac{1}{4}$, $\sin 2u = 2\sin u \cdot \cos u$

$$= 2 \cdot \frac{1}{4} \sqrt{1 - (\tfrac{1}{4})^2} = \frac{\sqrt{15}}{8}$$

when $\sin u = \frac{3}{4}$, $\sin 2u = 2 \cdot \frac{3}{4} \sqrt{1 - (\frac{3}{4})^2} = \frac{3\sqrt{7}}{8}$

$\longrightarrow$ final result $= 16(\arcsin\frac{3}{4} - \arcsin\frac{1}{4})$
$+ 16\left(\frac{\sqrt{15}}{8} - \frac{3\sqrt{7}}{8} \right)$

2. (a) $$\int \frac{2x}{(x+3)(3x+1)}\, dx = \int \frac{\frac{3(3x+1) - (x+3)}{4}}{(x+3)(3x+1)}\, dx$$

$$= \frac{1}{4} \int \left(\frac{3}{x+3} - \frac{1}{3x+1} \right) dx$$

$$= \frac{1}{4} \left(3\ln x - \frac{1}{3} \ln(3x+1) \right) + C$$

(b) $$\int \frac{x+3}{(x-1)^3}\, dx = \int \frac{(x-1)+4}{(x-1)^3}\, dx$$

$$= \int \frac{1}{(x-1)^2} + \frac{4}{(x-1)^3}\, dx = -(x-1)^{-1} - 2(x-1)^{-2} + C$$

(c) $\displaystyle\int_1^2 \frac{x^2+x+1}{x^2+x}\,dx = \int_1^2 1+\frac{1}{x^2+x}\,dx$

$$= \int_1^2 1+\frac{1}{x(x+1)}\,dx$$

$$\int \frac{1}{x(x+1)}\,dx = \int \frac{(x+1)-x}{x(x+1)}\,dx = \int \frac{1}{x}-\frac{1}{x+1}\,dx$$

$$= \ln x - \ln(x+1)$$

$$\int_1^2 \left(1+\frac{1}{x(x+1)}\right)dx = \left(x + \underbrace{\ln x - \ln(x+1)}_{=\ln\frac{x}{x+1}}\right)\Big|_1^2$$

$$= \left(2+\ln\frac{2}{3}\right) - \left(1+\ln\frac{1}{2}\right)$$

$$= 1+\ln\frac{4}{3}$$

(d) $\displaystyle\int_1^2 \frac{1}{x(x+3)}\,dx = \int_1^2 \frac{\frac{(x+3)-x}{3}}{x(x+3)}\,dx$

$$= \frac{1}{3}\int_1^2 \left(\frac{1}{x}-\frac{1}{x+3}\right)dx$$

$$= \frac{1}{3}\left(\ln x - \ln(x+3)\right)\Big|_1^2$$

$$= \frac{1}{3}\ln\frac{x}{x+3}\Big|_1^2 = \frac{1}{3}\left(\ln\frac{2}{5} - \ln\frac{1}{4}\right)$$

$$= \frac{1}{3}\ln\frac{8}{5}.$$