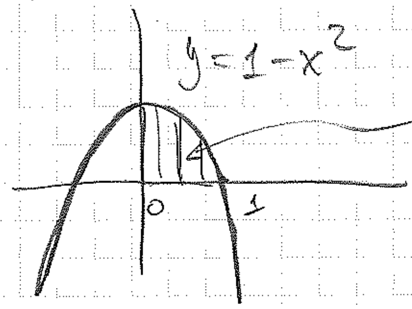
Solution to Practice Problems for MATH 104 Final

(spring 2018)

1.



$$A(x) = (1-x^2)^2$$

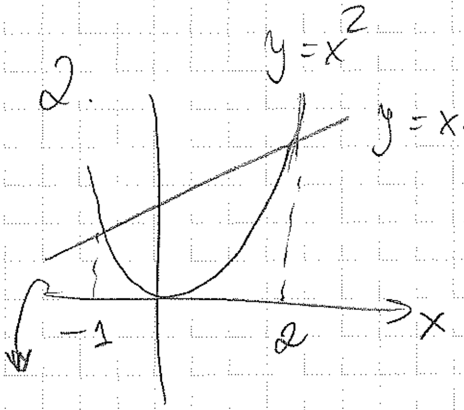
$$V = \int_0^1 A(x)\,dx = \int_0^1 (1-x^2)^2\,dx$$

$$= \int_0^1 1 - 2x^2 + x^4\,dx = 1 - \frac{2}{3} + \frac{1}{5}$$

$$= \frac{15-10+3}{15} = \frac{8}{15}$$

F

2.



$$x^2 = x+2 \iff x^2 - x - 2 = 0$$

$$\iff x = 2 \text{ or } x = -1$$

"Washer Method"

$$V = \pi\int_{-1}^{2} (x+2)^2 - x^4\,dx = \pi\int_{-1}^{2} x^2 + 4x + 4 - x^4\,dx$$

$$= \pi\left(\frac{x^3}{3} + 2x^2 + 4x - \frac{x^5}{5}\right)\Big|_{-1}^{2} = \frac{72}{5}\pi$$

C

3. $y = \dfrac{e^{2x}}{2} + \dfrac{e^{-2x}}{8}, \quad 0 \le x \le 2$

$$y' = e^{2x} - \frac{e^{-2x}}{4}$$

$$1 + (y')^2 = 1 + e^{4x} - \frac{1}{2} + \frac{e^{-4x}}{16} = e^{4x} + \frac{1}{2} + \frac{e^{-4x}}{16} = \left(e^{2x} + \frac{e^{-2x}}{4}\right)^2$$

$$\ell = \int_0^2 e^{2x} + \frac{e^{-2x}}{4}\,dx = \frac{e^{2x}}{2} - \frac{e^{-2x}}{8}\Big|_0^2 = \frac{e^4}{2} - \frac{e^{-4}}{8} - \underbrace{\left(\frac{1}{2} - \frac{1}{8}\right)}_{3/8}$$

C

4. $\begin{cases} x\dfrac{dy}{dx} = y + x^3 e^x \\ y(1) = 0 \end{cases}$

$$\frac{dy}{dx} - \frac{y}{x} = x^2 e^x, \quad \mu = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

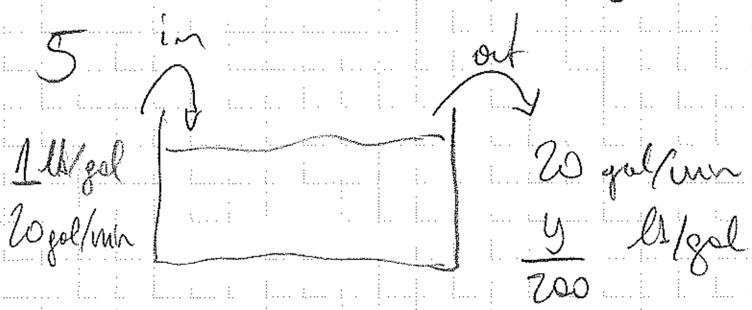
$$\underbrace{\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2}}_{\frac{d}{dx}\left(\frac{1}{x}y\right)} = xe^x \Rightarrow \frac{y}{x} = \int xe^x = xe^x - \int e^x = xe^x - e^x + C$$

$$\Rightarrow y = x^2e^x - xe^x + Cx$$

$$0 = y(1) = e - e + C \Rightarrow C = 0.$$

$$y(x) = x^2e^x - xe^x \Rightarrow y(3) = 9e^3 - 3e^3 = 6e^3. \quad \text{E}$$

5



$V_0 = 200$ gal

$$\begin{cases} \dfrac{dy}{dt} = 20 - 20\cdot\dfrac{y}{200} \\ y(0) = 50 \end{cases}$$

$$\frac{dy}{dt} + \frac{y}{10} = 20 \quad \mu = e^{\int \frac{1}{10}} = e^{\frac{t}{10}}$$

$$\underbrace{e^{t/10}\frac{dy}{dt} + \frac{y}{10}e^{t/10}}_{\frac{d}{dt}\left(e^{t/10}y\right)} = 20e^{t/10} \Rightarrow e^{t/10}\cdot y = 200e^{t/10} + C$$

$$\Rightarrow y = 200 + Ce^{-t/10}$$

$$50 = y(0) = 200 + C \Rightarrow C = -150$$

$$y(t) = 200 - 150e^{-t/10}$$

$$y(10\ln 3/2) = 200 - 150e^{-\ln 3/2} = 200 - 150\cdot\frac{2}{3} = 100 \quad \text{A}$$

6 $\displaystyle\int_0^{\pi/2} \cos^3 x \sin^2 x \, dx = \frac{1}{3}\sin^3\frac{\pi}{2} - \frac{1}{5}\sin^5\frac{\pi}{2} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$

A

$\displaystyle\int \cos^3 x \sin^2 x \, dx = \int (1-\sin^2 x)\sin^2 x \cdot \cos x \, dx$

$u = \sin x$

$du = \cos x \, dx$

$\displaystyle = \int (1-u^2)\, u^2 \cdot du = \int u^2 - u^4 \, du$

$\displaystyle = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

7. $\displaystyle\int_0^{\infty} x e^{-2x} = \lim_{b\to\infty} \int_0^b x e^{-2x} dx = \lim_{b\to\infty} \left.\frac{x e^{-2x}}{-2}\right|_0^b + \int_0^b \frac{e^{-2x}}{2}$

$\displaystyle = \lim_{b\to\infty}\left(-\frac{b e^{-2b}}{2} + \frac{e^{-2b}}{-4} - \frac{1}{-4}\right) = \frac{1}{4}$

D

8. $\displaystyle\frac{2x-2}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$

$A(x-3) + B(x+1) = 2x-2$

$x=3;\ 4B = 4 \Rightarrow B=1$

$x=-1;\ -4A = -4 \Rightarrow A = 1$

$\displaystyle I = \int_0^1 \frac{1}{x+1} + \frac{1}{x-3}\, dx = \ln|x+1| + \ln|x-3|\Big|_0^1 = 2\ln 2 - \ln 3$

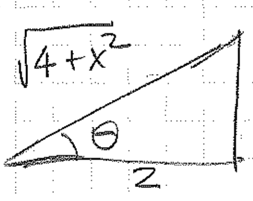
C

9. $\displaystyle\int \sqrt{4+x^2}\, dx = \int 2\sec\theta \cdot 2\sec^2\theta \, d\theta = 4\int \sec^3\theta \, d\theta$

$x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$

$\displaystyle = 4\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C\right)$



$\displaystyle = 2\,\frac{\sqrt{4+x^2}}{2}\cdot\frac{x}{2} + 2\ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$

$$= \frac{1}{2} x\sqrt{4+x^2} + 2 \ln\left|\frac{x}{2} + \frac{\sqrt{4+x^2}}{2}\right| + c$$

$$\int_0^4 \sqrt{4+x^2}\,dx = \frac{1}{2}\, 4\sqrt{20} + 2\ln\left|2+\frac{\sqrt{20}}{2}\right| - 2\cancel{\ln 1}^{\,0}$$

$$= 2 \cdot 2\sqrt{5} + 2\ln\left(2+\sqrt{5}\right) = \underline{4\sqrt{5} + 2\ln\left(2+\sqrt{5}\right)}.$$

Answer Key is wrong! $\to$ D

Appendix:

$$\int \sec^3 x\,dx = \sec x \tan x - \int \tan^2 x \cdot \sec x\,dx$$

$$\left|\begin{array}{ll} u = \sec x & du = \sec x \tan x\,dx \\ dv = \sec^2 x\,dx & v = \tan x \end{array}\right|$$

$$= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\Rightarrow 2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

$$\Rightarrow \int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C$$

10. $$\mu = \int_1^e x \ln x\,dx = \frac{x^2}{2}\ln x\Big|_1^e - \int_1^e \frac{x^2}{2}\cdot\frac{dx}{x} = \frac{e^2}{2} - \frac{1}{2}\,\frac{x^2}{2}\Big|_1^e$$

$$= \frac{e^2}{2} - \frac{1}{4}\left(e^2-1\right) = \frac{e^2}{4} + \frac{1}{4} = \frac{e^2+1}{4}$$ B

11.
$$A = \lim_{n\to\infty} \arctan\left(\ln(n^2+1)\right) = \frac{\pi}{2}$$

$$B = \lim_{n\to\infty} e^{e^{-n^2}} = 1$$

$$C = \lim_{n\to\infty} \frac{1}{1+\frac{1}{1+\frac{1}{n}}} = \frac{1}{1+1} = \frac{1}{2}$$

$$A+B+C = \frac{3}{2} + \frac{\pi}{2}$$

B

12. $A = \sum_{n=1}^{\infty} \frac{n \ln n}{n^3+1} < \infty$ b/c $\frac{\ln n}{n^2} \sim \frac{n \ln n}{n^3+1}$

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} -\frac{\ln x}{x}\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} -\frac{\ln b}{b} - \frac{1}{x}\Big|_1^b = \lim_{b\to\infty} -\frac{\ln b}{b} - \frac{1}{b} + 1 = \underline{1}.$$

$B = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} < \infty$ by Alternating Series Test

$C = \sum_{n=1}^{\infty} \frac{n^4}{2n!} < \infty$ $\lim_{n\to\infty} \frac{(n+1)^4}{\cancel{2}(n+1)!} \frac{\cancel{2}n!}{n^4} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^4 \frac{\cancel{n!}}{(n+1)\cancel{n!}} = 0$

All series converge E

13. $40 + 0.9 \cdot 40 + 0.9^2 \cdot 40 + \cdots = \sum_{n=0}^{\infty} (0.9)^n \cdot 40$

$$= \frac{40}{1-0.9} = \underline{\underline{400}} \quad \text{D}$$

14. $\sum_{n=1}^{\infty} \frac{(x+3)^n}{2^n\sqrt{n}}$ need $1 > \lim_{n\to\infty} \frac{(x+3)^{n+1}}{2^{n+1}\sqrt{n+1}} \frac{2^n\sqrt{n}}{(x+3)^n} = \lim_{n\to\infty} \frac{1}{2}\sqrt{\frac{n}{n+1}}\,|x+3|$

$$\Rightarrow \frac{1}{2}|x+3| < 1 \Rightarrow |x+3| < 2 \Rightarrow x \in (-5,-1)$$

$x=-5$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} < \infty$ $\quad x=-1$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$ $\quad [-5,1)$

B

15. $F'(x) = x^3 \cos 2x = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+3}}{(2n)!}$

$\Rightarrow F(x) = \int F'(x)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+4}}{(2n+4)(2n)!}$ A