Math 312, Spring 2016
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Homework Set 10

Due: Apr 22, 2016 (in class)

1. Bretscher Section 8.3: 4, 15, 20, 21, 24

2. Find an inner product \( \langle \cdot, \cdot \rangle \) on \( \mathbb{R}^2 \) such that the basis \( \{(-2, 3), (1, 4)\} \) is orthonormal.

3. Find an inner product \( \langle \cdot, \cdot \rangle \) on \( \mathbb{R}^3 \) such that the basis \( \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\} \) is orthonormal.

4. Let \( A \in M_{n \times n}(\mathbb{R}) \) be a nondegenerate symmetric matrix, so that \( \langle v, w \rangle = \langle Av, w \rangle = w^t Av \) is a (possibly indefinite) inner product. Show that \( T: \mathbb{R}^n \to \mathbb{R}^n \) is an isometry of this inner product, that is, \( \langle T(v), T(w) \rangle = \langle v, w \rangle \) if and only if \( T^t AT = A \).

   Note: If \( A = \text{Id} \) is the \( n \times n \) identity matrix, then this shows that \( T: \mathbb{R}^n \to \mathbb{R}^n \) is an Euclidean isometry if and only if \( T^t T = \text{Id} \), i.e., \( T \in \text{O}(n) \) is an orthogonal matrix.

5. Use the above criterion (Problem 4) to show that for all \( -1 < \beta < 1 \),

\[
T = \frac{1}{\sqrt{1 - \beta^2}} \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}
\]

is an isometry of Minkowski space \((\mathbb{R}^2, \eta)\), where \( \eta(x, y) = -x_1 y_1 + x_2 y_2 \) is the Lorentz inner product corresponding to the matrix \( A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \).

6. Use the above criterion (Problem 4) to find isometries of \((\mathbb{R}^2, \xi_\theta)\), where the Lorentz inner product \( \xi_\theta(x, y) = x^t(A_\theta)y \) corresponds to the matrix \( A_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \), in terms of the isometries \( T \) of the Minkowski space \((\mathbb{R}^2, \eta)\) given in Problem 5.

Hint: In Problem 5, you were dealing with the case \( \theta = \pi \), i.e., \( \eta = \xi_\pi \). To reduce this problem to the previous problem, begin by computing \( B = R_\alpha A_\theta R_\alpha^{-1} \), where \( R_\alpha = \begin{pmatrix} \sin \alpha & -\cos \alpha \\ \cos \alpha & \sin \alpha \end{pmatrix} \) is the matrix of counter-clockwise (Euclidean) rotation by \( \alpha \), and then choose a convenient \( \alpha \) in terms of \( \theta \) so that \( B = A_\pi \). Then use Problem 4.

Hint 2: \( \cos(a - b) = \cos a \cos b + \sin a \sin b \)
\( \sin(a - b) = \sin a \cos b - \sin b \cos a \).